8 Queens, Sudoku, and Projection Methods

Heinz Bauschke

Mathematics, UBC Okanagan

Research supported by NSERC and CRC Program

heinz.bauschke@ubc.ca

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The IRMACS Centre, SFU, Burnaby

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Overview

1. PROJECTION METHODS
2. 8 QUEENS AND SUDOKU
3. FURTHER APPLICATIONS
4. WORK IN PROGRESS
5. PAPERS AND OCANA

Based on:

- *joint works with P. Combettes (Paris VI) and R. Luke (Delaware);*
- *work in progress with J. Schaad (UBCO).*

1. Projection Methods

Basic Setup

- X is a real Hilbert space (or simply $\mathbb{R}^J$)
- $(x, y) \mapsto \langle x|y \rangle$ inner product
- $x \mapsto \|x\| = \sqrt{\langle x|x \rangle}$ norm
- A and B are closed subsets of X with $A \cap B \neq \varnothing$
- Projectors P_A and P_B
- Reflectors $R_A := 2P_A - \mathrm{Id}$ and $R_B := 2P_B - \mathrm{Id}$

The Feasibility Problem

Find a point in $A \cap B$.

A and B are the "constraints", with "solutions" $A \cap B$.

Projection methods are iterative algorithms that employ P_A, P_B, R_A, R_B to generate a sequence $(x_n)_{n \in \mathbb{N}}$ that (hopefully!) converges to a solution in $A \cap B$.

Many projection methods and **convergence results** exist when A and B are both **convex**.

A new constraint & solution

von Neumann and Lions-Mercier

von Neumann (1933): $x_0 \in X$,

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = P_B P_A x_n.$$

A and B linear $\Rightarrow x_n \to P_{A \cap B} x_0$;
A and B convex $\Rightarrow x_n \rightharpoonup \bar{x} \in A \cap B$ (Bregman, 1965).

Lions-Mercier (1979): $x_0 \in X$,

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \tfrac{1}{2} x_n + \tfrac{1}{2} R_B R_A x_n =: T x_n.$$

Assume that A and B are convex. Then $(x_n)_{n \in \mathbb{N}}$ converges weakly to some $\bar{x} = T\bar{x}$, and every weak cluster point of $(P_A x_n)_{n \in \mathbb{N}}$ lies in $A \cap B$.
$\dim X < +\infty \Rightarrow P_A x_n \to \bar{x} \in A \cap B$.

Remarks

- Very few nonconvex convergence results.
- Similar performance on **convex** feasibility problems.
- How to deal with $N > 2$ constraints $C_1, \ldots, C_N$? **Product Space Trick!** Work in X^N with the **two** sets

$$A := \left\{(x, \ldots, x) \in X^N \mid x \in X\right\}, \; B := C_1 \times \cdots \times C_N$$

and their projections

$$P_A(x_1, \ldots, x_N) = (\bar{x}, \ldots, \bar{x}), \text{ where } \bar{x} = \tfrac{1}{N} x_1 + \cdots \tfrac{1}{N} x_N,$$

and

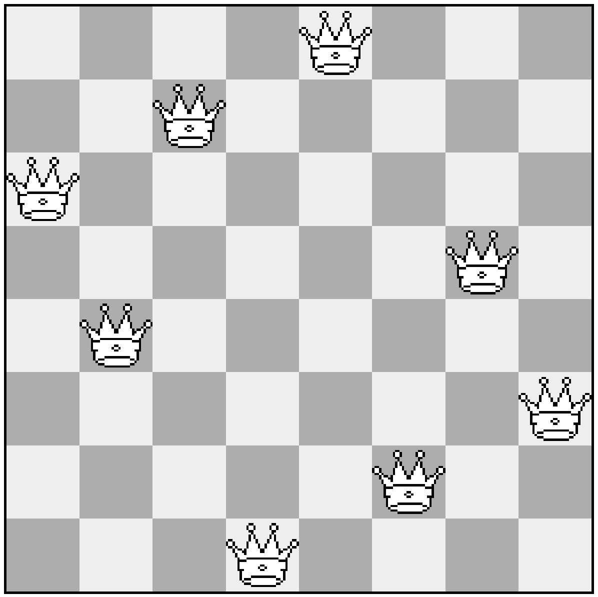
$$P_B(x_1, \ldots, x_N) = (P_{C_1} x_1, \ldots, P_{C_N} x_N).$$

3. 8 QUEENS AND SUDOKU

The 8 Queens Puzzle

Place 8 Queens on a chess board such that no two attack one another.

Recall that a Queen moves horizontally, vertically, or diagonally.



`http://www.stetson.edu/~efriedma/mathmagic/0201.html`.

Model

Solutions $\cong$ certain 8×8 matrices with $0-1$ entries.
Last solution corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \{0,1\}^{8\times 8}.$$

Space and Constraints

Underlying space is

$$\mathbb{R}^{8\times 8}.$$

There are 4 types of constraints.

- vertical: **exactly one** queen in every column.
- horizontal: **exactly one** queen in every row.
- **at most one** queen in every SW-NE diagonal.
- **at most one** queen in every NW-SE diagonal.

Corresponds to either **standard unit vectors** or **zero vectors**.

Projection

Lemma. Let $x \in \mathbb{R}^8$. Then the projection of x onto the set of standard unit vectors S is

$$\left\{e_i \in \mathbb{R}^8 \mid x_i = \max x_j\right\}.$$

Proof. Indeed,

$$\begin{aligned}
e_i \in P_S x &\Leftrightarrow (\forall j) \quad \|x - e_i\| \leq \|x - e_j\| \\
&\Leftrightarrow (\forall j) \quad \|x - e_i\|^2 \leq \|x - e_j\|^2 \\
&\Leftrightarrow (\forall j) \quad \|x\|^2 - \langle x|e_i\rangle + \|e_i\|^2 \leq \|x\|^2 - 2\,\langle x|e_j\rangle + \|e_j\|^2 \\
&\Leftrightarrow (\forall j) \quad \langle x|e_j\rangle \leq \langle x|e_i\rangle\,.
\end{aligned}$$

Projection Methods & 8 Queens

- All projectors (and hence reflectors) are easy to compute (use also tie-break rule).
- 4 types of constraint are dealt with in the *product space* $\mathbb{R}^{8\times 8} \times \mathbb{R}^{8\times 8} \times \mathbb{R}^{8\times 8} \times \mathbb{R}^{8\times 8} \equiv \mathbb{R}^{8\times 8\times 4} = \mathbb{R}^{256}$.
- Numerical results:
  - von Neumann fails miserably.
  - Lions-Mercier does well!
- Try it at `www.schaad.ca`
- Why Lions-Mercier does well is largely unexplained. (There is no reason that it should always work.)

Sudoku

Given a partially completed grid, fill it so that each column, each row, and each of the nine 3×3 regions contains the digits from 1 to 9 only once.

	7	5		9				6
	2	3		8			4	
8					3			1
5			7		2			
	4		8		6		2	
			9		1			3
9			4					7
	6			7		5	8	
7				1		3	9	

Tue May 6th, 2008 www.SudokuCentral.com

Solution of Example

1	7	5	2	9	4	8	3	6
6	2	3	1	8	7	9	4	5
8	9	4	5	6	3	2	7	1
5	1	9	7	3	2	4	6	8
3	4	7	8	5	6	1	2	9
2	8	6	9	4	1	7	5	3
9	3	8	4	2	5	6	1	7
4	6	1	3	7	9	5	8	2
7	5	2	6	1	8	3	9	4

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Model à la Veit Elser (Cornell)

Solutions $\cong$ certain $9 \times 9 \times 9$ cubes with $0 - 1$ entries.

Let's speak of "rows"/"columns" (in each "floor"), and "elevators": For instance, the $(1, 2)$ solution entry $\mathbf{7}$ corresponds to the 7^{th} standard unit vector in the $(1, 2)$ elevator (shaft).

Underlying space is

$$\mathbb{R}^{9\times9\times9}.$$

Constraints

- each column contains a permutation of $\{1, 2, \ldots, 9\} \cong$ for a given column, the corresponding floors are represented by standard unit vectors.
- each row contains a permutation of $\{1, 2, \ldots, 9\} \cong$ for a given row, the corresponding floors are represented by standard unit vectors.
- each 3×3 region contains a permutation of $\{1, 2, \ldots, 9\} \cong$ for a given region, the corresponding floors are represented by standard unit vectors.
- the pre-filled Sudoku entries correspond to already fixed standard unit vectors in elevators.

Projection Methods & Sudoku

Numerical experiments are similar to results for the 8 Queens Puzzle.

- All projectors (and hence reflectors) are easy to compute.
- 4 types of constraint $\Rightarrow$ we work in the *product space* $\mathbb{R}^{9\times9\times9}\times\mathbb{R}^{9\times9\times9}\times\mathbb{R}^{9\times9\times9}\times\mathbb{R}^{9\times9\times9} = \mathbb{R}^{9\times9\times9\times4} = \mathbb{R}^{2916}$.
- Numerical results:
  - von Neumann fails miserably (again)!
  - Lions-Mercier does well! Again, try it at `www.schaad.ca`
- Again, it is not clear why Lions-Mercier works.

3. Further Applications

Phase Retrieval

In a Hilbert space X (L_2 or Euclidean space), denote the *Fourier transform* of x by $\widehat{x}$ or $\mathcal{F}x$. The goal is to recover an unknown image x from a priori information and measured data:

- The *support* of x lies in a given set D, i.e.,

$$x = 0 \quad \text{on} \quad X \setminus D;$$

- The Fourier transform *modulus* m is observed, i.e.,

$$|\widehat{x}| = m;$$

- and possibly *nonnegativity*, i.e., $x \geq 0$.

Error Reduction (ER) Algorithm

$$
\begin{tikzcd}
x \arrow[r] & \boxed{\text{Force } x = 0 \text{ on } X \setminus D} \arrow[r] & x \arrow[d] \\
\boxed{\mathcal{F}^{-1}} \arrow[u] & & \boxed{\mathcal{F}} \arrow[d] \\
\widehat{x} \arrow[u] & \boxed{\text{Force } |\widehat{x}| = m} \arrow[l] & \widehat{x} \arrow[l]
\end{tikzcd}
$$

Details of (ER)

- Forcing $x = 0$ on $X \setminus D$ corresponds to a *projection* onto the (convex) vector subspace

$$S := \{x \in X : x = 0 \text{ outside D}\}.$$

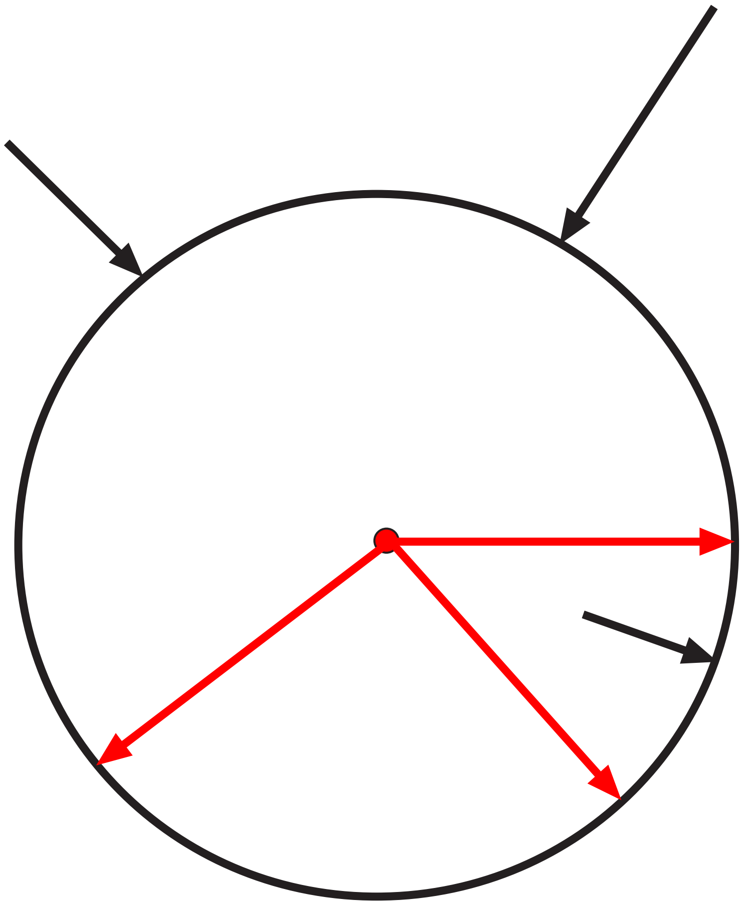
- In contrast, the set of images with prescribed Fourier magnitude

$$M := \{x \in X : |\widehat{x}| = m\}$$

  is *nonconvex*. However, the *set-valued* projection onto M is easy to compute explicitly.

$\Rightarrow$ (ER) algorithm $\cong$ von Neumann's method for S and M!
Phase Retrieval Problem $=$ Find $x \in S \cap M$!

Projection onto M...



...uniqueness is fine — except at the origin!

Fienup's (HIO) Algorithm

- The *Hybrid Input-Output (HIO)* algorithm updates

$$x_{n+1}(t) := \begin{cases} \big(P_M x_n\big)(t), & \text{if } t \in D; \\ x_n(t) - \big(P_M(x_n)\big)(t), & \text{otherwise.} \end{cases}$$

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- This can be rewritten as

$$x_{n+1} = \left(\tfrac{1}{2} R_S R_M + \tfrac{1}{2}\mathsf{Id}\right)(x_n),$$

where

$$R_C := 2P_C - \mathsf{Id}$$

is the *reflection* with respect to C.

Main Observations

- (ER) $\leftrightarrow$ "von Neumann" (Levi-Stark 1984).

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- (ER) $\leftrightarrow$ "von Neumann" (Levi-Stark 1984).
- (HIO) $\leftrightarrow$ "Lions-Mercier" (B-Combettes-Luke 2002).
- As done in practice and as the convex counterparts predict, the pertinent sequence to monitor is

$$(P_M x_n)_{n\in\mathbb{N}}.$$

Incorporating nonnegativity

- Replace the vector subspace S by the *convex cone*

$$S_+ := \{x \in X : x \geq 0, \text{ and } x = 0 \text{ on } X \setminus D\}.$$

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- Now Fienup's (HIO), which considered the state of the art, is

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- But this does *not* correspond to "Lions-Mercier"!!

(HPR) Algorithm

- The true counterpart to "Lions-Mercier", termed the *Hybrid Projection Reflection (HPR)* method, is

$$x_{n+1} = \left(\tfrac{1}{2} R_{S_+} R_M + \tfrac{1}{2}\mathsf{Id}\right)(x_n);$$

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- Numerical experiments (B-Combettes-Luke 2003) show that (HPR) outperforms (HIO)!

Lions-Mercier and Physics

Veit Elser (Physics, Cornell) has pioneered many applications of “Lions-Mercier”.

Warning: Different terminology in the physics community: Elser speaks of the **“Difference Map Algorithm”**, product space technique is referred to as **“Cloning”**, **“Replicas”**, and **“Divide and Concur”**.

Some of Elser's Applications

- Graph colouring
- Logical Satisfiability ("3-SAT problem")
- Protein folding
- Sphere packing (new, higher-density packings!)

While "Lions-Mercier" is considerably slower than special-purpose algorithms for each of these problems, it does a surprisingly good job in finding solutions.

4. WORK IN PROGRESS

On the unreasonable efficiency

- Study simple planar cases; e.g., A is a line and B contains finitely many points ($\varnothing \neq A \cap B \subsetneq B$).
  - B contains 2 elements $\Rightarrow$ von Neumann will fail yet Lions-Mercier will **always** work.
  - B contains 3 elements $\Rightarrow$ Lions-Mercier does fail in certain cases (“How rare?”).
- Duality Theory seems to suggest that Lions-Mercier has a certain “self-dual” property.
- Consider the case when B is the disjoint union of two convex sets — the “inconsistent” convergence results of B-Combettes-Luke should apply.
- Have fun with other applications!

5. PAPERS AND OCANA

Papers

- HHB, P.L. Combettes, and D.R. Luke: “Finding best approximation pairs relative to closed convex sets in Hilbert spaces”, *Journal of Approximation Theory* 127 (2004), pp. 178–192.
- HHB and J. Schaad: Web applications for the 8 Queens Puzzle and for Sudoku, 2008, `www.schaad.ca`
- V. Elser, I. Rankenburg, and P. Thibault: “Searching with iterated maps”, *Proceedings of the National Academy of Sciences* 104 (2007), pp. 418–423.
- S. Gravel and V. Elser: “Divide and concur: A general approach to constraint satisfaction”, 2008, `http://arxiv.org/abs/0801.0222v1`

OCANA

The **OCANA** — "**O**ptimization, **C**onvex **A**nalysis and **N**onsmooth **A**nalysis" — **Research Group** at UBC Okanagan in Kelowna, B.C. consists of:

- Heinz Bauschke (Convex Analysis, Math)
- Yong Gao (Discrete Optimization, Comp Sci)
- Donovan Hare (Discrete Optimization, Math)
- **NEW:** Warren Hare (Continuous Optimization, Math)
- Patricia Lasserre (Image Processing, Comp Sci)
- Yves Lucet (Numerical Optimization, Comp Sci)
- Shawn Wang (Nonsmooth Analysis, Math)

UBC Okanagan

Campus View

OCANA (cont'd)

- **Graduate Program: MSc** & **PhD** in **Mathematics** or **Interdisciplinary Graduate Studies**. Current Graduate Students with Optimization Theme: V. Koch (Fall 08), S. Moffat, J. Schaad, and L. Yao.
- **OCANA Seminar Series:** 20+ talks so far. Speakers from: Dal, SFU, UBCO, UBCV, UVic, BC Cancer Centre, the U.S., France, and Japan.
- **West Coast Optimization Meeting:** Sept. 6, 2008. Speakers include: M. Théra (Limoges), T. Rockafellar (UWash), **S. Simons** (UCSB), ...

`http://people.ok.ubc.ca/bauschke/ocana.html`

Kelowna — Okanagan Lake

Floating Bridge