Quartic $\mathbb{Q}$-derived polynomials with distinct roots

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Introduction

Let k be a field. A k-derived polynomial is a polynomial $f \in k[x]$ of degree $n \geq 1$ such that $f^{(0)}$, $f^{(1)}$, ..., $f^{(n-1)}$ split completely in $k[x]$.

- e.g. $k = \mathbb{Q}$

$$\begin{aligned} f &= x(x-1) \\ f' &= 2x-1 \end{aligned}$$

- e.g. $k = \mathbb{Q}$

$$\begin{aligned} f &= x(x-9)(x-24) \\ f' &= 3(x-4)(x-18) \\ f'' &= 6(x-11) \end{aligned}$$

- If $f \in k[x]$ is k-derived, then the transformation $x = \alpha y + \beta$ (where $\alpha \in k^*, \beta \in k$) produces a k-derived polynomial in $k[y]$.

- If $f \in k[x]$ is k-derived, then $g = \lambda f$ (where $\lambda \in k^*$) is a k-derived polynomial in $k[x]$.

For the most part, we will be interested in k-derived polynomials modulo these transformations.

Let $D_k(n)$ denote the set of k-derived polynomials of degree $n \geq 1$, up to the equivalence above.

Also note that if $f \in k[x]$ is k-derived, then f' is k-derived.

- For each degree $n \geq 1$ and partition of $n = n_1 + \ldots + n_t$ into $n_1 \geq \ldots \geq n_t \geq 1$, we say a polynomial $f \in k[x]$ which splits completely in $k[x]$ is of type $(n_1, \ldots, n_t)$ if it has t roots of respective multiplicities $n_1, \ldots, n_t$.

- e.g. The following is a $\mathbb{Q}$-derived polynomial of type $(2, 1, 1)$.

$$f = x^2 (x-1) \left(x - \frac{49710}{167167} \right)$$
$$f' = 4x \left(x - \frac{1164}{1837} \right) \left(x - \frac{855}{364} \right)$$
$$f'' = 12 \left(x - \frac{342}{1169} \right) \left(x - \frac{485}{286} \right)$$

Buchholz and MacDougall made a comprehensive study of $\mathbb{Q}$-derived polynomials.

- $D_{\mathbb{Q}}(1) = \{x\}$
- $D_{\mathbb{Q}}(2) = \left\{x^2, x(x-1)\right\}$
- $D_{\mathbb{Q}}(3) = \left\{x^3\right\}$
  $\bigcup\left\{x(x-1)(x-a) \mid a = \frac{w(w-2)}{w^2-1}, w \in \mathbb{Q}\right\}$

To analyze $D_{\mathbb{Q}}(n)$ for $n = 4$, we break up the problem into types.

- $\mathbb{Q}$-derived polynomials of type $(2,1,1)$ are given by $x^2(x-1)(x-a)$ where

$$a = \frac{9(2w+z-12)(w+2)}{(z-w-18)(8w+z)} \neq 0, 1$$

  and $(w,z) \in E(\mathbb{Q})$. The elliptic curve E is given by $z^2 = w(w-6)(w+18)$. E has rank 1 with generator $(w,z) = (-2, 16)$. Its conductor is 576.

- $\mathbb{Q}$-derived polynomials of type $(2,2)$ are given by $x^2(x-1)^2$.

- $\mathbb{Q}$-derived polynomials of type $(3,1)$ are given by $x^3(x-1)$.

- $\mathbb{Q}$-derived polynomials of type (4) are given by x^4.

Conjecture. *(Buchholz and MacDougall) There are no* $\mathbb{Q}$*-derived polynomials of type* $(1,1,1,1)$.

Theorem. *(Buchholz and MacDougall, Flynn) If Conjecture 1 holds, then* $D_{\mathbb{Q}}(n) = \left\{x^n, x^{n-1}(x-1)\right\}$ *for all* $n \geq 5$.

k-derived polynomials of type $(1, 1, 1, 1)$

Main Difficulty: These correspond to k-rational points on a complicated surface, whereas the cases dealt with Buchholz and MacDougall, Flynn involved studying the k-rational points on curves.

To give a sense of how complicated this surface is, here is one approach to setting up the equations by Buchholz and MacDougall.

- We can scale coordinates so one of the roots of f is 1.

$$\begin{aligned} f &= (x-1)(x-a)(x-b)(x-c) \\ &= x^4 - \sigma_1 x^3 + \sigma_2 x^2 - \sigma_3 x + \sigma_4 \\ f' &= 4x^3 - 3\sigma_1 x^2 + 2\sigma_2 x - \sigma_3 \\ f'' &= 12x^2 - 6\sigma_1 x + 2\sigma_2 \end{aligned}$$

- We can translate coordinates so the one of the roots of f' is 0 so we have $\sigma_3 = 0$ which is equivalent to

$$c = \frac{-ab}{ab+a+b}.$$

- For f to be k-derived, the discriminant of f'/x and f'' must be squares in k. These two constraints do not change the dimension of the problem.

- The constraint from the two discriminants being a square lead to a pair of quartic equations

$$u^2 = r_4 b^4 - r_3 b^3 + r_2 b^2 + r_1 b + r_0$$
$$v^2 = s_4 b^4 - s_3 b^3 + s_2 b^2 + s_1 b + s_0$$

where the $r_i, s_j \in k[a]$.

- The k-rational points on the surface defined by the above equations correspond to quartic k-derived polynomials.

- The quartic k-derived polynomials of type $(1,1,1,1)$ corresponds to the k-rational points of an open subset of the above surface.

- If we only impose the condition that f' splits completely, or that f'' splits completely, we only have to deal with one pencil of cubic equations.

- So in general, there are many examples of f such that f' splits completely or f'' splits completely. These can be obtained by find a value for a such that the fiber over it is an elliptic curve with high rank.

- Here is just one example:

$$f = (x + 9817)(x + 1307)(x - 2741)(x - 8383)$$
$$f' = 4(x - 648)(x + 6931)(x - 6283)$$

Another Way.

- The original problem is too complicated to tackle head on IMHO. We need an easier problem, but not too easier.

- Consider the less stringent diophantine problem of finding quartic polynomials $f \in k[x]$ with two linear factors and a quadratic factor, and whose derivative f' splits completely.

- This is motivated by the example

$$f = x^4 - 1 = (x-1)(x+1)(x^2+1)$$
$$f' = 4x^3.$$

- Without loss of generality, we can translate coordinates so f has a root at 0.

$$\begin{aligned} f &= x(x-c)(x^2+ax+b) \\ f' &= 4x^3+3(-c+a)x^2+2(-ca+b)x-cb \\ f'' &= 12x^2+6(-c+a)x-2ca+2b \end{aligned}$$

- Let T denote the surface in $\mathbb{A}^7$ in the variables $(a, b, c, \alpha, \beta, \gamma, \lambda)$ defined by the equations resulting from the constraints that

$$f(x) = x(x-c)(x^2+ax+b)$$
$$f'(x) = \lambda(x-\alpha)(x-\beta)(x-\gamma)$$
$$\Delta(f) = b^2c^2(c^2+ca+b)^2(a^2-4b)^2 \neq 0.$$

- The surface T is quasi-affine and its k-rational points correspond to quartic polynomials $f \in k[x]$ with two linear factors (normalized so one of them is x), distinct roots, and whose derivative f' splits completely in $k[x]$.

We wish to give a nicer model for T.

Theorem. *(Schultz) A cubic polynomial in $\mathbb{Q}[x]$ splits completely if and only if it has a root in $\mathbb{Q}$ and its discriminant is a square in $\mathbb{Q}$.*

Theorem.

$$\begin{aligned} C :&(-\beta + c + a)Y^2 - 4a^3 - 12ca^2 \\ &+ 3\beta^2 a - 28c^2 a + \beta^3 - 20c^3 + 3\beta^2 c \\ E_1 :&a^3 + 3\, ca^2 + 7\, c^2 a + 5\, c^3 - 3\, \beta^2 a - 3\, \beta^2 c + 2\, \beta^3 \\ E_2 :&y_2^2 = 256 - 1536 b_2 + 3072 b_2^2 - 2048 b_2^3. \end{aligned}$$

Then T is isomorphic to the open subset of a closed set in $\mathbb{P}^3$. The closed set is a union of three components $C \bigcup A \bigcup B$, where C is a cubic surface in $\mathbb{P}^3$, $A \cong E_1$, and $B \cong E_2 \times \mathbb{P}^1$ are elliptic curves in $\mathbb{P}^2$.

- To get the surface S classifying k-derived polynomials of type $(1,1,1,1)$, we need to impose two more conditions that the quadratic factors of f and f'' split in k.

- This is equivalent to requiring the discriminants of f and f'' to be squares in k. In odd characteristic, this can be explicitly achieved by requiring two further conditions:

$$\begin{aligned} a^2 - 4b &= u^2 \\ 12(3a^2 + 3c^2 + 2ac - 8b) &= v^2 \end{aligned}$$

- This shows that the surface S is isomorphic to an open subset of $\tilde{C} \bigcup \tilde{A} \bigcup \tilde{B}$, where $\tilde{C}$, $\tilde{A}$, $\tilde{B}$ are covers of C, $A \cong E_1$, $B \cong E_2 \times \mathbb{P}^1$ of degree 4.

The cubic surface C.

To understand T, we need to understand C, E_1, and $E_2 \times \mathbb{P}^1$. In this section, we focus our attention on the component of T contained in C.

- The cubic surface C is singular. The singular locus consists of the following three double points:

$$P_C = (1, 0, 0, -2)$$
$$P_A = (1, 0, -3, 1)$$
$$P_B = (1, 0, 3, 1)$$

- Let's translate coordinates so $P_C = (1, 0, 0, 0)$, $P_A = (1, 0, -3, 3)$, $P_B = (1, 0, 3, 3)$.

- There is a birational map $\phi : C \dashrightarrow \mathbb{P}^2$ given by $(a, c, y, \beta) \mapsto (c, y, \beta)$, namely projection to the $a = 0$ plane.

- The inverse $\psi : \mathbb{P}^2 \dashrightarrow C$ of ϕ is defined by drawing the line L through P_C and $(0, c, y, \beta)$. The line L will intersect C at one other point which is the value of ψ on $(c, y, \beta) \in \mathbb{P}^2$.

- Thus, the problem of finding $f \in k[x]$ with two linear factors and a quadratic factor and such that f' splits completely is diophantine problem which is birational to $\mathbb{P}^2$.

- Explicitly, such polynomials are obtained as

$$f = x(x-c)(x^2+ax+b)$$

where for $(c, y, \beta) \in \mathbb{P}^2$ we put

$$\begin{aligned}
c &= c \\
a &= \frac{-y^2\beta + y^2c + \beta^3 - 20c^3 + 3\beta^2 c}{H(c,y,\beta)} \\
b &= \frac{(-y+\beta-2c)(y+\beta-2c)(44c^2+20c\beta+3\beta^2-3y^2)}{32H(c,y,\beta)} \\
H &= 28c^2 - 3y^2 + 3\beta^2 + 12c\beta
\end{aligned}$$

- If we don't mind that students have to solve a quadratic equation, then the birational map ψ gives a parametrization of some nice quartic examples.

- Note that the birational map $\psi : \mathbb{P}^2 \dashrightarrow C$ is not defined when $H(c, y, \beta) \neq 0$. When $H(c, y, \beta) = 0$, this corresponds to four lines in C which are not in the image of the birational map ψ.

- To be complete in getting all nice quartic examples, we should also consider the other components of T lying in E_1 and $E_2 \times \mathbb{P}^1$.

According to classical results on (singular) cubic surfaces, C should have 12 lines on it (cf. Bruce and Wall for a modern reference).

One can explicitly determine all 12 lines (previous talk).

The Big Picture

Let X be a variety defined over $\mathbb{Q}$. The Zariski closure of the union of all images of non-trivial rational maps $A \dashrightarrow X$, where A is an abelian variety, is called the special subset Σ_X of X.

Conjecture. *(Bombieri-Lang) $X(k) - \Sigma_X(k)$ is finite for every number field k.*

- We want to determine Σ_S for our surface S.

- One approach might be to study the cubic surface C in some detail and then go up to S.

A Piece of the Big Picture

- To understand the surface C, a standard thing to do is determine its Picard group and intersection pairing on divisors.

- The cubic surface C lies in $\mathbb{P}^3$. Consider the birational map $\phi : C \dashrightarrow \mathbb{P}^2$ given by $(a, c, y, \beta) \mapsto (c, y, \beta)$.

- From the theory of monoidal transformations, C is isomorphic to a sequence of blow-ups and their inverses of $\mathbb{P}^2$.

- Getting this sequence of blow ups and their inverses explicitly would allow one to compute the Picard group of C or at least of a nonsingular model $\bar{C}$.

- Recall the universal property of blowing-up.

- Let $f : X \to S$ be a birational morphism of nonsingular surfaces and suppose the rational map f^{-1} is undefined at a point p of S. Then f factorizes as

$$f : X \xrightarrow{g} \widehat{S} \xrightarrow{\widehat{\epsilon}} S$$

  where g is a birational morphism and $\widehat{\epsilon}$ is the blow-up of S at p.

- As initially given, ϕ is defined except at the point $P_C = (1, 0, 0, 0)$.

- Let $\bar{C}$ be the blow-up of C at the singular points P_A, P_B, P_C.

Theorem. *$\bar{C}$ is non-singular and ϕ extends to a birational morphism $\bar{\phi} : \bar{C} \to \mathbb{P}^2$.*

- By explicitly computing the graph of $\bar{\phi}$, we see that $\bar{\phi}^{-1}$ is undefined at the four points $(0, \pm 1, 1), (-\frac{1}{2}, \pm\frac{2}{\sqrt{3}}, 1)$.

- Let $\bar{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ at these 4 points. Then $\bar{\phi}$ factors through a birational morphism $\bar{\rho} : \bar{C} \to \bar{\mathbb{P}}^2$.

- Again, by explicitly computing the graph of $\bar{\rho}$, we see that $\bar{\rho}^{-1}$ is undefined at two points U^+, U^- on $\bar{\mathbb{P}}^2$.

- Let $\tilde{\mathbb{P}}^2$ be the blow-up of $\bar{\mathbb{P}}^2$ at U^+, U^-. Then $\bar{\rho}$ factors through a birational morphism $\bar{\theta} : \bar{C} \to \tilde{\mathbb{P}}^2$.

Theorem. *The map* $\bar{\theta} : \bar{C} \to \tilde{\mathbb{P}}^2$ *is an isomorphism.*

Corollary. $\mathrm{Pic}(\bar{C}) \cong \mathbb{Z}^7$

What next?

- Use this to determine Σ_S.
- Even better. Prove something about $S(\mathbb{Q}) - \Sigma_S(\mathbb{Q})$.

To give some idea of the computations involved:

Explicitly, the graph of ϕ consists of the points $(a, c, y, \beta) \times (u, v, w) \in C \times \mathbb{P}^2$ satisfying

$$cw - \beta u = 0$$
$$yw - \beta v = 0$$
$$\beta\left(v^2w - w^3 - v^2u - 3w^2u + 20u^3\right) + aw\left(28u^2 - 3v^2 + 3w^2 + 12uw\right) = 0$$

when $w \neq 0$.

We need more equations on the other charts because of the extra component $\beta = w = 0$ included in the first set of equations.

$$20cu^2 - yuv + \beta(-3uw - w^2 + v^2) + a(28u^2 - 3v^2 + 3w^2 + 12uw) = 0$$
$$y(20u^3 - v^2u) + v\beta(-3uw - w^2 + v^2)$$
$$+ av(28u^2 - 3v^2 + 3w^2 + 12uw) = 0$$
$$yw - \beta v = 0$$
$$\beta(v^2w - w^3 - v^2u - 3w^2u + 20u^3) + aw(28u^2 - 3v^2 + 3w^2 + 12uw) = 0$$

when $u \neq 0$.

$$cv - yu = 0$$
$$y(20u^3 - v^2u) + v\beta(-3uw - w^2 + v^2)$$
$$+ av(28u^2 - 3v^2 + 3w^2 + 12uw) = 0$$
$$yw - \beta v = 0$$
$$\beta(v^2w - w^3 - v^2u - 3w^2u + 20u^3) + aw(28u^2 - 3v^2 + 3w^2 + 12uw) = 0$$

when $v \neq 0$.