

Motivation

Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

Results

Statement

Outline of proof

Examples

# The Riemann zeta function and asymptotics for Stieltjes fractions

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# Riemann zeta function

## ► Motivation

### Introduction

Stieltjes transform & continued fraction

Real  $J$ -fractions & orthogonal polynomials

### Results

Statement

Outline of proof

### Examples

## Definition

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

## Continued fractions?

- Explicit CF expansion for  $\zeta(2)$  and  $\zeta(3)$ .
- Expansion for  $\zeta(3)$  arises in relation to Apéry's proof of irrationality.
- No interesting general formula.

▷ **Motivation**

**Introduction**

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

**Results**

Statement

Outline of proof

**Examples**

## The famous $\zeta(3)$ expansion...

$$\zeta(3) = \frac{6}{d(0) - \frac{1^6}{d(1) - \frac{2^6}{d(2) - \frac{3^6}{d(3) - \ddots}}}}$$

where

$$d(n) = 34n^3 + 51n^2 + 27n + 5$$



# Connection with polylogarithms

## ▷ Motivation

### Introduction

Stieltjes transform & continued fraction

Real  $J$ -fractions & orthogonal polynomials

### Results

Statement

Outline of proof

### Examples

## Definition

$$\text{Li}_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}$$

## Zeta function is special case

$$\text{Li}_\nu(1) = \zeta(\nu), \quad \text{Li}_\nu(-1) = (2^{1-\nu} - 1)\zeta(\nu)$$



## Stieltjes transform

$G(z)$  is the Stieltjes transform of a function  $v(t)$  if

$$G(z) = \int_0^\infty \frac{v(t)}{z+t} dt$$

where  $v(t)$  is non-negative.

## $\text{Li}_\nu$

Cvijović & Klinowski (1997) found that

$$-\text{Li}_n(-1/z) = \frac{1}{(n-1)!} \int_0^1 \frac{\log(1/t)^{n-1}}{z+t} dt$$

for  $n = 1, 2, \dots$



# (Modified) Stieltjes fraction (MSF)

## ▷ Motivation

### Introduction

Stieltjes transform & continued fraction

Real  $J$ -fractions & orthogonal polynomials

### Results

Statement

Outline of proof

### Examples

For each Stieltjes transform  $G(z)$  there exist unique  $b_k > 0$  such that

$$G(z) \sim \frac{b_1}{z + \frac{b_2}{1 + \frac{b_3}{z + \frac{b_4}{1 + \ddots}}}}$$

for  $z \rightarrow \infty$ .



# Continued fraction for $\zeta(n)$

## > Motivation

### Introduction

Stieltjes transform & continued fraction

Real  $J$ -fractions & orthogonal polynomials

### Results

Statement

Outline of proof

### Examples

Putting everything together yields

$$\zeta(n) = \frac{1}{(1 - 2^{1-n})} \mathop{K}_{k=1}^{\infty} \frac{b_k(n)}{1}$$

where the coefficients  $b_k(n)$

- ▶ can be written down in terms of Hankel determinants,
- ▶ or can be computed using the qd-algorithm.

...

What is  $\lim_{k \rightarrow \infty} b_k(n)$  ?

▷ **Motivation**

**Introduction**

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

**Results**

Statement

Outline of proof

**Examples**

## A few values of $b_k(n)$

$n \setminus k$	1	2	3	4	5
1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{5}$
2	1	$\frac{1}{4}$	$\frac{7}{36}$	$\frac{17}{63}$	$\frac{647}{2975}$
3	1	$\frac{1}{8}$	$\frac{37}{216}$	$\frac{217}{999}$	$\frac{30271}{143375}$
4	1	$\frac{1}{16}$	$\frac{175}{1296}$	$\frac{493}{2835}$	$\frac{2081687}{10784375}$
5	1	$\frac{1}{32}$	$\frac{781}{7776}$	$\frac{26281}{189783}$	$\frac{10916749081}{64142065625}$



# The Stieltjes transform

Let  $v(t)$  be a non-negative weight function on  $(a, b) \subset (0, \infty)$  such that its moments

$$c_k = \int_a^b t^k v(t) dt$$

exist and are finite. The **Stieltjes transform**  $G(z)$  of  $v(t)$  is defined as

$$G(z) = \int_a^b \frac{v(t)}{z+t} dt$$

for  $-z \notin [a, b]$ .



# Correspondence at $z = \infty$

$G(z)$  admits an asymptotic expansion

$$G(z) \approx \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{z^{k+1}}$$

as  $z \rightarrow \infty$ .

There exists a unique modified Stieltjes fraction  $\hat{G}(z)$  which **corresponds** to  $G(z)$  at  $z = \infty$ , which means that the  $n$ th approximant  $g_n(z)$  has a Laurent series of the form

$$g_n(z) = \sum_{k=0}^{n-1} (-1)^k \frac{c_k}{z^{k+1}} + O(z^{-n-1}).$$



# Real $J$ -fractions (RJF)

- ▶ These are continued fractions of the form

$$\frac{\lambda_1}{z + l_1 - \frac{\lambda_2}{z + l_2 - \frac{\lambda_3}{z + l_3 - \ddots}}}$$

where  $\lambda_j > 0$  for  $j = 2, 3, \dots$  and  $l_j$  are real.

- ▶ For each real  $J$ -fraction there exists a unique formal Laurent series (FLS) which corresponds to it at  $z = \infty$ .



# Orthogonal polynomials

Motivation

Introduction

Stieltjes transform &  
continued fraction

▷ **Real  $J$ -fractions &  
orthogonal  
polynomials**

Results

Statement

Outline of proof

Examples

Let  $Q_n(z)$  denote the  $n$ th denominator of a real  $J$ -fraction, then

- ▶ they satisfy the recurrence relation

$$Q_n(z) = (z + l_n)Q_{n-1}(z) - \lambda_n Q_{n-2}(z),$$

- ▶ if the coefficients in the FLS are the moments of a weight function  $\nu(t)$ , then  $\{Q_n\}$  forms an orthogonal polynomial sequence with respect to this weight.



# Main result

Let  $v(t)$  be a non-negative weight function on  $(a, b)$  and let  $b_k$  denote the coefficients in the modified Stieltjes fraction which corresponds to the Stieltjes transform of  $v(t)$ .

- ▶ If  $v(t) > 0$  a.e. on  $(a, b)$ , then

$$\lim_{k \rightarrow \infty} b_{2k+1} = \frac{(\sqrt{b} + \sqrt{a})^2}{4},$$

$$\lim_{k \rightarrow \infty} b_{2k} = \frac{(\sqrt{b} - \sqrt{a})^2}{4}.$$

## Motivation

## Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

## Results

### ▷ **Statement**

Outline of proof

## Examples

- If  $a > 0$  and  $v(t) = (b - t)^\gamma (t - a)^\delta h(t)$  where  $\gamma, \delta > -1$  and  $h(t) > 0$  is analytic on  $[a, b]$ , then

$$b_{2k+1} \sim \frac{(\sqrt{b} + \sqrt{a})^2}{4} + \sum_{i=2}^{\infty} \frac{B_i^{(o)}}{k^i},$$

$$b_{2k} \sim \frac{(\sqrt{b} - \sqrt{a})^2}{4} + \sum_{i=2}^{\infty} \frac{B_i^{(e)}}{k^i},$$

for  $k \rightarrow \infty$ .

- If  $a = 0$  and  $v(t) = (\sqrt{b} - \sqrt{t})^\gamma (\sqrt{b} + \sqrt{t})^\delta h(\sqrt{t})/\sqrt{t}$  where  $\gamma, \delta > -1$  and  $h(t) > 0$  is analytic on  $[-\sqrt{b}, \sqrt{b}]$ , then

$$b_k \sim \frac{b}{4} + \sum_{i=2}^{\infty} \frac{B_i}{k^i}, \quad k \rightarrow \infty.$$

Motivation

Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

Results

Statement

▷ *Outline of proof*

Examples



# Outline of proof

1. Relate MSF & RJF using even contraction and equivalence transformations, to yield

$$\lambda_n(v) = b_{2n-2}b_{2n-1}, \quad l_n(v) = -(b_{2n-1} + b_{2n}).$$

2. “Uncouple” coefficients  $b_k$  using appropriate transformation.
3. Use asymptotic behaviour of OP and recurrence coefficients.

Case  $(a, b) = (0, \infty)$  due to Jones and Van Assche (1998).



## Sequence of transformations in 2.

Motivation

Introduction

Stieltjes transform & continued fraction

Real  $J$ -fractions & orthogonal polynomials

Results

Statement

► *Outline of proof*

Examples

weight	support	$\lambda_n$	$l_n$
$v(t)$	$(a, b)$	$b_{2n-2}b_{2n-1}$	$-(b_{2n-1} + b_{2n})$
$bv(bt)$	$(\frac{a}{b}, 1)$	$\frac{b_{2n-2}b_{2n-1}}{b}$	$-\frac{b_{2n-1}+b_{2n}}{b}$
$ t bv(bt^2)$	$(-1, -\sqrt{\frac{a}{b}}) \cup (\sqrt{\frac{a}{b}}, 1)$	$\frac{b_n}{b}$	0

- If  $a = 0$  then use asymptotic behaviour of recurrence coefficients on  $(-1, 1)$ ,
- else map  $(-1, -\sqrt{\frac{a}{b}}) \cup (\sqrt{\frac{a}{b}}, 1)$  to  $(-1, 1)$  using Geronimo and Van Assche (1988).

Motivation

Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

Results

Statement

▷ *Outline of proof*

Examples



Map  $(-1, -\sqrt{\frac{a}{b}}) \cup (\sqrt{\frac{a}{b}}, 1)$  to  $(-1, 1)$

Put

$$v_2(t) = |t|bv(bt^2), \quad v_3(t) = \frac{b-a}{2}v\left(\frac{b-a}{2}t + \frac{b+a}{2}\right),$$

then

$$\lambda_{2n+1}(v_2) = -\frac{b-a}{2b} \sqrt{\lambda_{n+1}(v_3)} \frac{p_{n-1}\left(-\frac{b+a}{b-a}; v_3\right)}{p_n\left(-\frac{b+a}{b-a}; v_3\right)},$$

$$\lambda_{2n}(v_2) = -\frac{b-a}{2b} \sqrt{\lambda_{n+1}(v_3)} \frac{p_n\left(-\frac{b+a}{b-a}; v_3\right)}{p_{n-1}\left(-\frac{b+a}{b-a}; v_3\right)}.$$



## General

If  $v(t) > 0$  a.e. then

$$\lambda_n(v) \rightarrow \frac{1}{4},$$
$$\frac{p_n(x; v)}{p_{n-1}(x; v)} \rightarrow x + \sqrt{x^2 - 1},$$

(Rakhmanov, 1977 & 1983).

## Detailed expansion

Use Riemann-Hilbert techniques to get additional terms (Kuijlaars, McLaughlin, Van Assche and Vanlessen, 2003).



# Example 1: hypergeometric function

It can be shown that if  $c > a$ , then (Euler)

$$\frac{1}{z} {}_2F_1 \left( a, 1; c; -\frac{1}{z} \right) = \int_0^1 \frac{v(t)}{z+t} dt,$$

where

$$v(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} t^{a-1} (1-t)^{-a+c-1}$$

and that

$$b_{2k+2} = \frac{(a+k)(c+k-1)}{(c+2k-1)(c+2k)},$$

$$b_{2k+1} = \frac{k(c-a+k-1)}{(c+2k-2)(c+2k-1)},$$

from which it follows that

$$b_{2k+2} = \frac{1}{4} + \left( \frac{a}{4} - \frac{1}{8} \right) \frac{1}{k} + O(k^{-2}),$$

$$b_{2k+1} = \frac{1}{4} - \left( \frac{a}{4} - \frac{1}{8} \right) \frac{1}{k} + O(k^{-2}).$$

If  $a = 1/2$  then

$$v(t) = \frac{\Gamma(c)}{\sqrt{\pi} \Gamma(c - 1/2)} \frac{(1-t)^{c-3/2}}{\sqrt{t}}$$

which satisfies the conditions for the detailed asymptotic expansion.



## Example 2: square root

It can be shown that

$$\frac{1}{\sqrt{(z+a)(z+b)}} = \int_a^b \frac{v(t)}{z+t} dt,$$

where

$$v(t) = \frac{1}{\pi\sqrt{(t-a)(b-t)}}.$$

Some computations yield

$$c_k = \int_a^b t^k v(t) dt = (\sqrt{ab})^k P_k \left( \frac{b+a}{2\sqrt{ab}} \right)$$

where  $P_k(x)$  is the Legendre polynomial of degree  $k$ .

## Motivation

## Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

## Results

Statement

Outline of proof

## ► Examples

Taking  $a = 1/2$  and  $b = 1$  and using the qd-algorithm gives

$b_{2k+1}$	$b_{2k}$
7.5000000000000000e - 01	4.166666666666667e - 02
7.083333333333333e - 01	2.20588235294118e - 02
7.27941176470588e - 01	2.14646464646465e - 02
7.28535353535353e - 01	2.14471403812825e - 02
7.28552859618718e - 01	2.14466250371692e - 02
7.28553374962831e - 01	2.14466098668435e - 02
7.28553390133156e - 01	2.14466094202708e - 02
7.28553390579729e - 01	2.14466094071250e - 02
7.28553390592875e - 01	2.14466094067380e - 02

## Motivation

## Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

## Results

Statement

Outline of proof

## ► Examples

- Logarithmic convergence?!

- **Always read the small print!!**

Kuijlaars et. al. say:

*It turns out that for the case  $4\gamma^2 - 1 = 4\delta^2 - 1 = 0$ , all coefficients  $B_i^{(o)}$  and  $B_i^{(e)}$  vanish. In that case one can prove that*

$$b_{2k+1} \sim \frac{(\sqrt{b} + \sqrt{a})^2}{4} + O(e^{-ck}),$$

$$b_{2k} \sim \frac{(\sqrt{b} - \sqrt{a})^2}{4} + O(e^{-ck}).$$

## Motivation

## Introduction

Stieltjes transform &  
continued fraction

Real  $J$ -fractions &  
orthogonal polynomials

## Results

Statement

Outline of proof

## ► *Examples*



# So how about $\zeta(n)$ ?

►  $b_k \rightarrow 1/4$

► Need asymptotics for polynomials orthogonal with respect to

$$\tilde{v}(t) = |t| \log(1/t^2)^{n-1}$$

on  $[-1, 1]$ .

► Riemann-Hilbert?