A history of, and recent results on Beta-expansion

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(Allouche, Borwein, Frougny, Garth, Tweedle)

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History - Counting systems over the ages

- **Base 2**
  Binary, useful for computers
- **Base 3**
  Setun computers, Moscow State University, 1958-1970
- **Base 5**
  Some aboriginal languages from Australia, Western Africa, South America
- **Base 8**
  Yuki of Northern California
- **Base 10**
  Modern

History - Counting systems over the ages (ii)

- **Base 11**
  Message hidden in π base 11, (*Contact*, Sagan)
- **Base 15**
  Used for telephone routing over IP
  Huli of Papua New Guinea
- **Base 16**
  Computers
- **Base 20**
  Mayan, Central/Western Africa
- **Base 60**
  Babalyonian

Example - Integer Base

Example (Base 10)

$$\begin{aligned} \pi &= 3.14159\cdots_{(10)} \\ &= 3\cdot 10^{0} + 1\cdot 10^{-1} + 4\cdot 10^{-2} + 1\cdot 10^{-3} + \cdots \\ &= \sum_{n\geq 0} a_n (10)^{-n} \end{aligned}$$

with $a_n \in \{0, 1, \cdots, 9\}$.

Definition - Base β representation

Definition

Let S be a finite set of Digits and $|\beta| > 1$.
We say x has a *base β representation* if

$$x = \sum_{n=-k}^{\infty} a_n \beta^{-n}$$

where $a_n \in S$, and $k \in \mathbb{Z}$.

Remark

- Historically, $\beta \in \mathbb{N}$, $S = \{0, 1, \cdots, \beta - 1\}$.
- β can be negative, real, complex.
- S does not need to be a subset of the integers.

Example - Balanced base 3 representation

Example

- Consider $\beta = 3$, and $S = \{-1, 0, 1\}$.
- This is called the *balanced base 3 representation*

$$\begin{aligned}
11_{(10)} &= (1)(1)(-1)_3 \\
12_{(10)} &= (1)(1)(0)_3 \\
13_{(10)} &= (1)(1)(1)_3 \\
14_{(10)} &= (1)(-1)(-1)(-1)_3
\end{aligned}$$

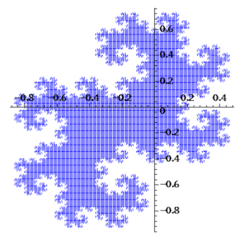
Remark

- Balanced base B representations are useful for high precision computing.

Example - Complex Base

Example

- Consider $\beta = -1 + i$, and $S = \{0, 1\}$.
- Consider all numbers of the form $0.a_1 a_2 a_3 \cdots_{(-1+i)}$.



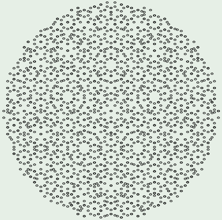
Remark

- This has a fractal boundary

Example - Complex Digit Set

Example

- Consider $\beta = \frac{1+\sqrt{5}}{2}$, and $S = \{0, 1, \omega, \omega^2, \omega^3, \omega^4\}$.
- Consider all "integers" less than 10 in absolute value.



Remark

- Useful in the study of quasicrystals

Definition - Beta-Expansions

Definition

Let $\beta > 1$ with $\beta \in \mathbb{R}$, and $S = \{0, 1, \cdots, \lceil \beta \rceil - 1\}$.
Then

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}$$

is said to be a *beta-expansion* of x.
This is represented as the infinite sequence $a_1 a_2 a_3 \cdots$.

Example

- Let $\beta = 10$, then the beta-expansion for 1 is $999 \cdots = 9^\omega$.
- Let $\beta = 2$, then the beta-expansion for 1 is $111 \cdots = 1^\omega$.

Example - Non-integer base

Example (Base $1.618\cdots$)

Consider β the Golden ratio, $\beta \approx 1.618$, $\beta^2 - \beta - 1 = 0$. Then

$$\begin{aligned} 1 &= \frac{1}{\beta} + \frac{1}{\beta^2} && \text{or} \\ &= \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{1}{\beta^4} && \text{or} \\ &= \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{1}{\beta^5} + \frac{1}{\beta^7} + \frac{1}{\beta^9} + \cdots \end{aligned}$$

So the beta-expansions of 1 include

$$110^\omega = 11,\ \ 1011,\ \ (10)^\omega$$

Results - Number of representations

Results

- If β is an integer, then 1 will have exactly one beta-expansion.
- If β is the golden ratio, ≈ 1.618, then 1 will have infinitely many beta-expansions.
- **Erdős, Joo, Komornik (1994)**
  For any $1 \leq N \leq \infty$, there exists an uncountable number of $\beta \in [1, 2]$ where 1 has exactly N expansions base β.

Definition - Greedy/Lazy-Expansions

Definition

- If $a_1 a_2 a_3 \cdots$ is the maximal beta-expansion for x (lexigraphically) then we say that $a_1 a_2 a_3 \cdots$ is a *greedy beta-expansion* for x.
- If $a_1 a_2 a_3 \cdots$ is the minimal beta-expansion for x (lexigraphically) then we say that $a_1 a_2 a_3 \cdots$ is a *lazy beta-expansion* for x.

Example

Consider β the Golden ratio, $\beta \approx 1.618$, where $\beta^2 - \beta - 1 = 0$.

- The greedy beta-expansion for 1 is 11.
- The lazy beta-expansion for 1 is 01^ω.

Results - Rényi

Results (Rényi 1957)

- First to study representations with non-integer base
- Studied greedy expansions only.
- Showed the “digits” cannot be chosen independently

Example

- Let $\beta = 1.618\cdots$ the greater root of $x^2 - x - 1$.
- The greedy expansion cannot contain the substring “011”.
- Assume the greedy expansion of some x contains “011”.
- Replace the substring “011” with “100”.
  - New expansion is an equally valid beta-expansion
  - New expansion is lexigraphically bigger
- Contradicts the expansion being greedy.

Algorithm - Computing the Greedy Expansion

Algorithm (Greedy Expansion)

- Define $T_\beta(x)$ as $T_\beta(x) = \beta \cdot x \pmod 1$
- Set $r_0 := x$. Recursively set $r_n = T_\beta(r_{n-1})$
- Set $a_n = \lfloor \beta \cdot r_{n-1} \rfloor$.
- Then $a_1 a_2 a_3 \cdots$ is the greedy expansion of x.

Example (beta-expansion of 1 base $\sqrt{2}$)

$$\begin{array}{ll} r_0 = 1 & a_1 = \lfloor \sqrt{2} \cdot 1 \rfloor = 1 \\ r_1 = \sqrt{2} - 1 & a_2 = \lfloor \sqrt{2} \cdot (\sqrt{2} - 1) \rfloor = 0 \\ r_2 = 2 - \sqrt{2} & a_3 = \lfloor \sqrt{2} \cdot (2 - \sqrt{2}) \rfloor = 0 \\ r_3 = 2\sqrt{2} - 2 & a_4 = \lfloor \sqrt{2} \cdot (2\sqrt{2} - 1) \rfloor = 1 \end{array}$$

This gives a beta-expansion of $1001\cdots$

Definition - Univoque numbers

Definition

If 1 has exactly 1 beta-expansion base β, then β is called a *univoque number*.

Example

Minimal Polynomial	Pisot Number	Greedy Expansion	Lazy Expansion
$x^4 - x^3 - 1$	1.380	1001	0001^ω
$x^4 - 2x^3 + x - 1$	1.866	111001	1101^ω
$x^4 - x^3 - 2x^2 + 1$	1.905	$11(10)^\omega$	$11(10)^\omega$
$x^4 - x^3 - x^2 - x - 1$	1.927	1111	$1(1101)^\omega$

Results - Univoque Numbers

Results

- **Daroczy, Katai, (1995)**
  The set of univoque numbers in (1,2) has Lebesgue measure 0.
- **Komornik and Loreti (1998)**
  There is a smallest univoque number $\kappa \approx 1.787231\cdots$.
- **Allouche and Cosnard (2000)**
  The value κ is transcendental.
- **Komornik, Loreti and Pethő (2003)**
  The value κ is a limit point of univoque numbers.

Definition - Companion Polynomials

Definition

Consider a beta-expansion of 1, say $a_1a_2a_3a_4\cdots$. Define $P_j(x) = x^j - a_1x^{j-1} - \cdots - a_j$.

Definition

Assume the beta-expansion is finite: $a_1a_2\cdots a_k000\cdots$,
or eventually periodic: $a_1a_2\cdots a_k(a_{k+1}\cdots a_n)^\omega$.
Define the *companion polynomial* as

$$R(x) = \begin{cases} P_k(x) & \text{if finite} \\ P_n(x) - P_k(x) & \text{if eventually periodic} \end{cases}$$

Remarks - Companion Polynomials

Remark

- We have $R(\beta) = 0$, but $R(x)$ need not be irreducible.
- This other factor is called the *co-factor*.

Corollary

If 1 *has a finite, or eventually periodic beta-expansion, then* β *is an algebraic integer.*

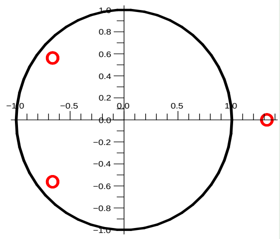
Definition - Pisot number

Definition

A *Pisot number* is a real algebraic integer > 1 such that all of it's conjugates are strictly less than 1 in modulus.

Example

Roots of $x^3 - x - 1$



Results - Pisot Numbers

Results

- **Salem (1963)**
  The set of Pisot numbers is closed, and the smallest Pisot number is $1.324\cdots$, the real root of $x^3 - x - 1$.
- **Amara (1966)**
  Describes all limit points.
- **Boyd (1978, 1984, 1985)**
  Gave an algorithm to compute all Pisot numbers in a fixed interval

Results - Structure of Pisot Numbers

Theorem (Amara 1966)

The limit points of the Pisot numbers in $(1, 2)$ *are:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \cdots < 2$$

where

- *The minimal polynomial of* φ_r *is* $\Phi_r(x) = x^{r+1} - 2x^r + x - 1$
- *The minimal polynomial of* ψ_r *is* $\Psi_r(x) = x^{r+1} - x^r - \cdots - 1$
- *The minimal polynomial of* χ *is* $\mathcal{X}(x) = x^4 - x^3 - 2x^2 + 1.$

Examples - Limit points of Pisot numbers

Examples

- $\varphi_1 = \psi_1 \approx$ 1.6180, the root of $x^2 - x - 1$
- $\varphi_2 \approx$ 1.7549, the root of $x^3 - 2x^2 + x - 1$
- $\psi_2 \approx$ 1.8393, the root of $x^3 - x^2 - x - 1$
- $\varphi_3 \approx$ 1.8668, the of $x^4 - 2x^3 + x - 1$
- $\chi \approx$ 1.9052, the root of $x^4 - x^3 - 2x^2 + 1$
- $\psi_3 \approx$ 1.9276, the root of $x^4 - x^3 - x^2 - x - 1$

Results - Pisot numbers and beta-expansions

Results

- **Bertrand (1977)**
  If β is a Pisot number, $x \in \mathbb{Q}(\beta)$, then x has a finite or periodic beta-expansion.
- **Schmidt (1980)**
  If all rational $0 < x < 1$ have periodic or finite beta-expansions, then β is a Pisot or Salem number.
- **Boyd (1996)**
  Showed the co-factor of the companion polynomial of a Pisot number need not be cyclotomic.
- **Allouche, Frougny, H., (2006)**
  Detailed analysis of univoque beta-expansions of 1 by Pisot numbers.

Results - Univoque Pisot numbers

Results (Allouche, Frougny, H., 2006)

- The smallest limit point of univoque Pisot numbers is $\chi \approx 1.90516$, the root of $x^4 - x^3 - 2x^2 + 1$.
- There are exactly two univoque Pisot numbers less than χ,
  - $1.880000\cdots$ the root of $x^{14} - 2x^{13} + x^{11} - x^{10} - x^7 + x^6 - x^4 + x^3 - x + 1$ univoque expansion $111001011(1001010)^\omega$.
  - $1.886681\cdots$ the root of $x^{12} - 2x^{11} + x^{10} - 2x^9 + x^8 - x^3 + x^2 - x + 1$ univoque expansion $111001101(1100)^\omega$
- The greedy and lazy expansions of all Pisot numbers less than χ have been determined.

Results - Beta-expansions of limit points

Theorem (Allouche, Frougny, H., 2006)

The only limit of Pisot numbers less than 2 that is univoque is χ.

Proof.

Pisot Number	Greedy expansion	Lazy expansion	Comment
φ_r	$1^r0^{r-1}1$	$1^{r-1}01^\omega$	
ψ_r	1^{r+1}	$(1^r0)^\omega$	
χ	$11(10)^\omega$	$11(10)^\omega$	Univoque

□

Definition - Regular Pisot Numbers

Definition (Regular Pisot Numbers)

Limit Points	Defining polynomials
φ_r	$\Phi_r(x)x^n \pm (x^r - x^{r-1} + 1)$
	$\Phi_r(x)x^n \pm (x^r - x + 1)$
	$\Phi_r(x)x^n \pm (x^r + 1)(x - 1)$
ψ_r	$\Psi_r(x)x^n \pm (x^{r+1} - 1)$
	$\Psi_r(x)x^n \pm (x^r - 1)/(x - 1)$
χ	$\mathcal{X}(x)x^n \pm (x^3 + x^2 - x - 1)$
	$\mathcal{X}(x)x^n \pm (x^4 - x^2 + 1)$

Results - The limit point χ

Theorem (Allouche, Frougny, H., 2006)

The value χ is a limit point of univoque Pisot numbers.

Proof.

The beta-expansion of the Pisot number satisfying

$$\mathcal{X}(x)x^{2k} - (x^3 + x^2 - x - 1),$$

with $k \geq 2$, is

$$111(01)^{k-2}1011((10)^{k-2}0111(01)^{k-2}1000)^{\omega},$$

which is univoque. ☐

Theorem (Allouche, Frougny, H., 2006)

No regular Pisot numbers less than χ are univoque.

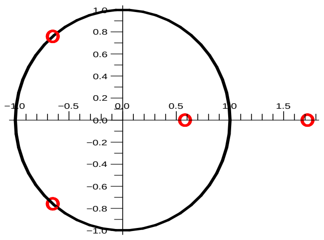
Definition - Salem Numbers

Definition

A *Salem number* is a real algebraic integer > 1 with conjugates ≤ 1 in modulus, and at least one conjugate is 1 in modulus.

Example

The roots of $x^4 - x^3 - x^2 - x + 1$.



Results - Salem Numbers

Results

- **D. H. Lehmer (1933)**
  Smallest known Salem number is $1.1762\cdots$, the root of $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$.
- **Boyd (1980, 1989), Mossinghoff (1998)**
  Numerous search for better Salem numbers, none found.
- Every Pisot number is a two sided limit point of Salem numbers.

Results - Salem Numbers (ii)

Results

- **Boyd (1989)**
  If β is a Salem number of degree 4, then the beta-expansion of 1 is eventually periodic.
- **Boyd (Conjecture)**
  If β is a Salem number of degree strictly greater than 6, the beta-expansions of 1 need not be eventually periodic.
- **H., Tweedle (2008)**
  Analysis of beta-expansions of Salem numbers approaching Pisot numbers.

Results - Structure of Salem Numbers

Definition

Let $p(x) = a_n x^n + \cdots + a_0$. We define the *reciprocal polynomial* as $p^*(x) = a_0 x^n + \cdots + a_n$.

Theorem

Let $p(x)$ be the minimal polynomial of a Pisot number q. Then for sufficiently large n, $p(x)x^n \pm p^*(x)$ has a root α_n that is a Salem number. Moreover $\alpha_n \to q$ as $n \to \infty$.

Results - Greedy expansions of Salem numbers

Theorem (H., Tweedle, 2008)

Let q be a Pisot number with minimal polynomial $p(x)$. Further let q have a "nice" finite beta-expansion, say $a_1 a_2 \cdots a_k$. Let α_n be the Salem number satisfying $p(x)x^n + p^*(x)$. Then for sufficiently large n, beta-expansion with base α_n is

$$1(a_2 a_3 .. a_k 0^{n-k-1} a_k a_{k-1} .. a_2 00)^\omega .$$

Example - Greedy expansion of Salem numbers

Examples

Minimal Polynomial	Beta Expansion
$x^2 - x - 1$	11
$(x^2 - x - 1)x^3 + (x^2 - x - 1)^*$	$1(1100)^\omega$
$(x^2 - x - 1)x^4 + (x^2 - x - 1)^*$	$1(10100)^\omega$
$(x^2 - x - 1)x^5 + (x^2 - x - 1)^*$	$1(100100)^\omega$
$(x^2 - x - 1)x^m + (x^2 - x - 1)^*$	$1(10^{m-3}100)^\omega$

Results - “Nice” Pisot numbers

Remark

- There exists infinite families of “nice” Pisot numbers.
- There are 3704 Pisot numbers, degree ≤ 30, in $(1, 2)$, of these, over 60% are “nice”

Definition - Beta-integers

Definition

We say x is a *beta-integer* if $x = \sum_{i=0}^{k} a_i \beta^i$ with a_i chosen using the greedy algorithm. Denoted $\mathbb{Z}_\beta$.

Remark

- Lazy expansions are never finite.
- For degree ≥ 3 is a difference between *all* expansions, and *greedy* expansions.
- Many applications to Quasi-crystals.

Definition - Spectra

Definition

Let $S \subset \mathbb{Z}$. Define the *spectra* as

$$\Lambda^S(q) = \{a_n q^n + \cdots + a_0 \mid n \in \mathbb{N}, a_i \in S\}$$

Remark

- For quadratic q, $\Lambda^{\{0,1\}}(q)$ is just the set of all beta-integers.
- For degree ≥ 3, $\Lambda^{\{0,1\}}(q)$ contains all the beta-integers.
- We don't need to restrict ourselves to $S \subset \mathbb{Z}$.

Definition - $\ell(q)$

Definition

We define $\ell(q) = \liminf |y - z|$ where the infimum is taken over all $y, z \in \Lambda^{\{0,1\}}(q)$, with $y \neq z$.

Remark

- $\ell(q) = \inf |y|$ where $y \in \Lambda^{\{0,\pm1\}}(q)$ and $y \neq 0$.

Results

- **Erdős, Joó, Joó (1992)**
  If q is the Pisot root of $q^n - q^{n-1} - \cdots - q - 1$ then $\ell(q) = \frac{1}{q}$.
- **Bugeaud (1996)**
  If $q \in (1, 2)$ does not satisfy a height 1 polynomial, then $\ell(q) = 0$

Results - $\ell(q)$

Results

- **Komornik, Loreti, Pedicini (2000)**
  If q is the Golden ratio, then $\ell^m(q) = |F_k\, q - F_{k+1}|$, where F_k is the k^{th} Fibonacci number.
- **Conjecture**
  If $q \in (1, 2)$ then $\ell(q) > 0$ if and only if q is a Pisot number.
- **Borwein, H. (2001)**
  An algorithm to calculate $\ell(q)$ and related values, assuming $\ell(q) > 0$.
- **H. (2003), Garth, H. (2006)**
  An algorithm to calculate frequency of gap sizes and related results
- **Stankov (to appear)**
  An algorithm to calculate $\ell(q)$ and related values, unconditional.

Results - Spectra (ii)

Theorem (Stankov, to appear)

If q is an algebraic integer, with conjugate q_2, and there exists a height 1 polynomial $p(x)$ with $|p(q)| \leq B_1$ and $|p(q_2)| > B_2$, then $\ell(q) = 0$.

Remark

- The bounds B_1 and B_2 are effectively computable
- This results is sufficiently “if and only if”-ish that an algorithm can be given which will calculate $\ell(q)$, unconditionally.

Open Questions/Future work

Open Questions/Future Work

- Do there exists Salem numbers with non-periodic, non-finite beta-expansions?
- Give a full classification of beta-expansions of regular Pisot numbers.
- Many results can be redone/reexamined for Lazy expansions.
- A precise understanding of the difference between beta-integers and $\Lambda^{\{0,1\}}(q)$ is needed.
- Does there exists a $q \in (1,2)$ with $\ell(q) > 0$ and q not Pisot.