Three Subset Sum Questions

G: Any group
$A, B \subseteq G$
$A + B = \{a + b \mid a \in A, b \in B\}$.

1. **Kneser** $|A + B| \geq f(|A|, |B|, \ldots)$
2. **Powerset Sums** $|\{\sum C \mid C \subseteq A\}| \geq f(|A|, \ldots)$
3. **Erdös/Ginsberg/Ziv** $k \geq f(|A|, \ldots)$ implies $0 \in \left\{\sum C \mid C \in \binom{A}{k}\right\}$

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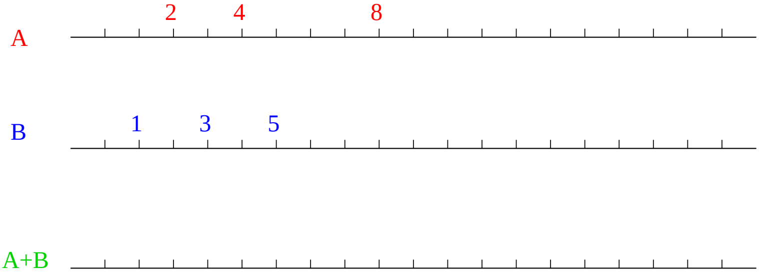

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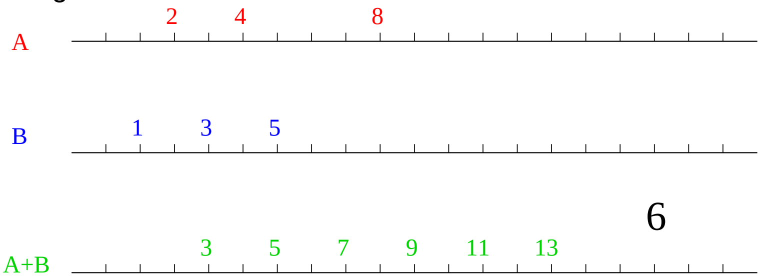

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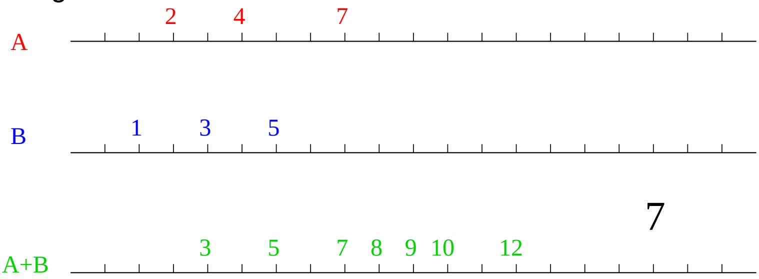

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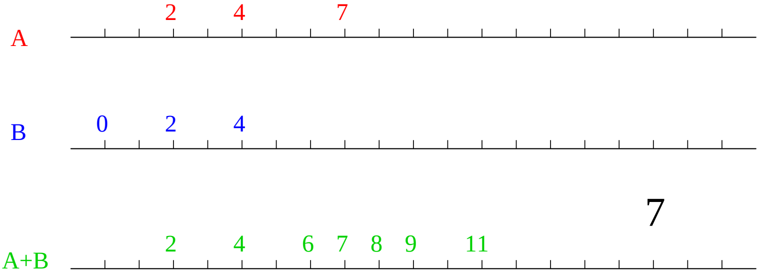

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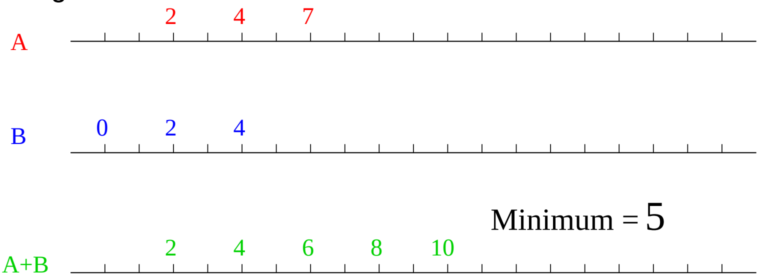

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Theorem

For integers,
$|A+B| \geq |A| + |B| - 1.$
Equality iff they are a.p. with common difference.

Kneser: $|A+B| \geq f(|A|,|B|,\ldots)$

Theorem (Cauchy-Davenport)

For $\mathbb{Z}/p\mathbb{Z}$, p prime,
$|A+B| \geq \min\{|A|+|B|-1, p\}$.

Kneser: $|A+B| \geq f(|A|, |B|, \ldots)$

Theorem

For $A, B \subseteq G$ Abelian,
$|A+B| \geq ???$

Mod 12: $\{2, 6, 10\} + \{3, 7, 11\} = \{1, 5, 9\}$

Kneser: $|A+B| \geq f(|A|,|B|,\dots)$

Theorem

For $A, B \subseteq G$ Abelian,
$|A+B| \geq \max\{|A|,|B|\}$ This is a joke :-)

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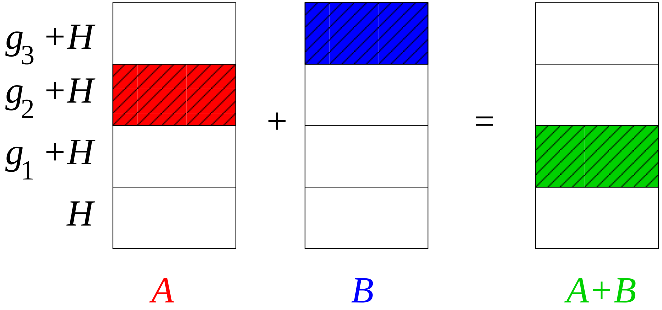

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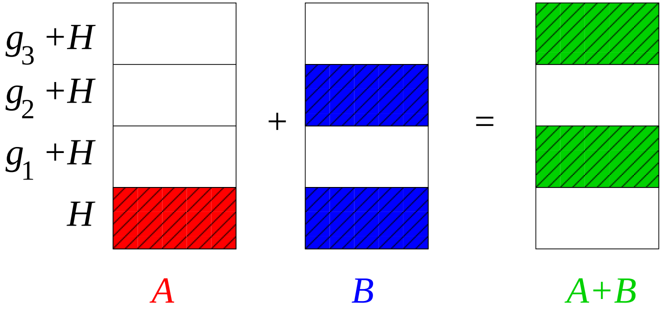

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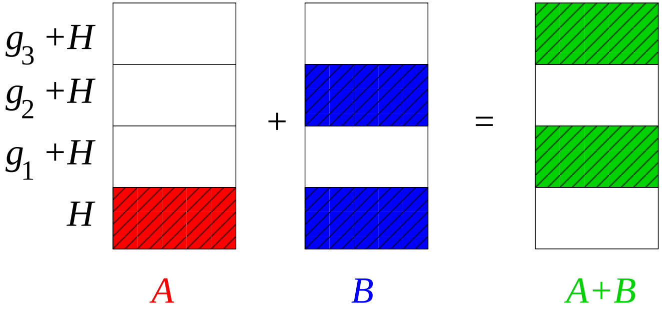
Theorem

For $A, B \subseteq G$ Abelian,
$|A+B| \geq f(|A|,|B|,H)$



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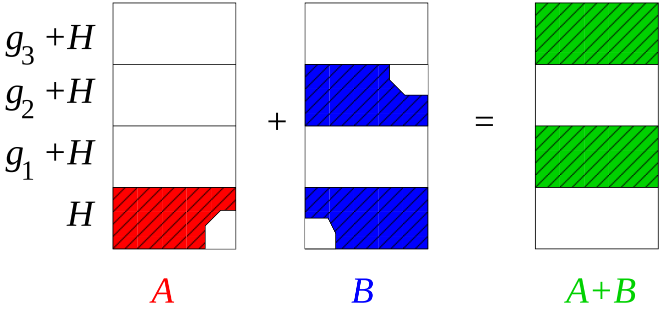
Theorem

For $A, B \subseteq G$ Abelian,
$|A+B| \geq f(|A|,|B|,H)$ must peek at $A+B$ to determine H



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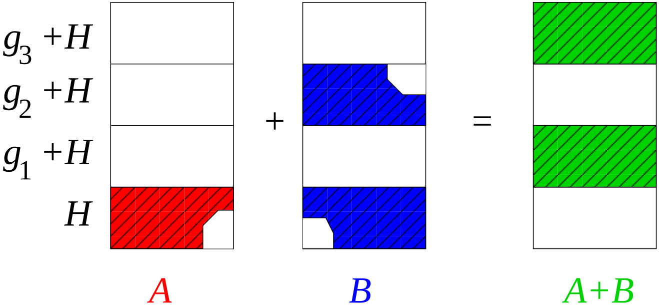
Theorem (Kneser)

For $A, B \subseteq G$ Abelian, where $H = \mathrm{Stab}(A+B)$,
$|A+B| \geq |A+H| + |B+H| - |H|$



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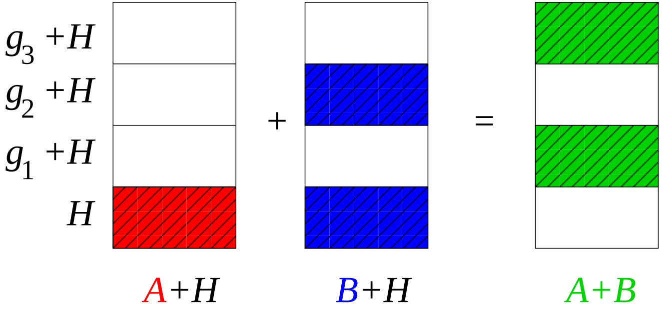
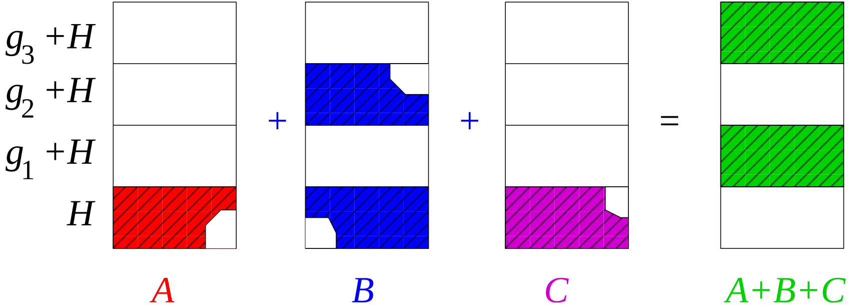


Iterated Kneser

Theorem

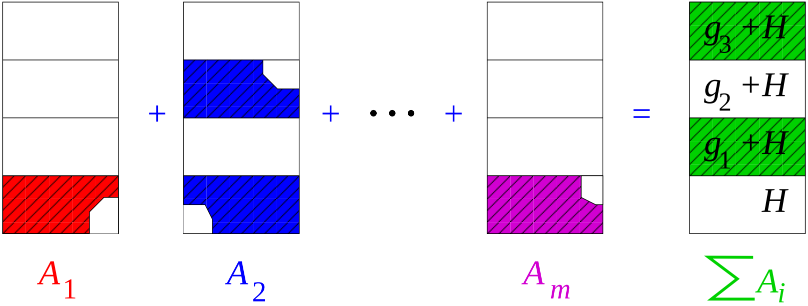
For $A, B, C \subseteq G$ Abelian, where $H = \mathrm{Stab}(A + B + C)$,
$|A + B + C| \geq |A + H| + |B + H| + |C + H| - 2|H|$



Iterated Kneser

Theorem (Iterated Kneser)

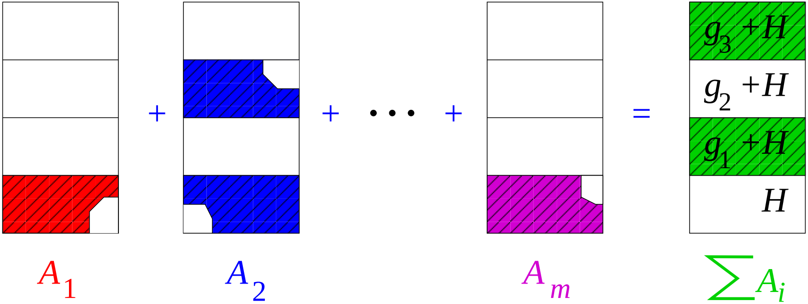
For $A_1, \ldots, A_m \subseteq G$ Abelian, where $H = \mathrm{Stab}(\sum_i A_i)$
$|\sum_i A_i| \geq \sum_i |A_i + H| - (m-1)|H|$



Iterated Kneser

Theorem (Iterated Kneser)

For $A_1, \ldots, A_k \subseteq G$ where $H = \mathrm{Stab}(\sum_i A_i)$
$|\sum_i A_i| \geq |H|$ *(total # represented H-cosets $-$ # additions)*



Powerset sums

$$\sum(A) := \left\{ \sum B \mid B \subseteq A \right\}$$

$$A = \{1, 3, 4\}$$
$$\sum(A) = \{0, 1, 3, 4, 5, 7, 8\}$$

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Powerset sums

Theorem (Easy)

If $A \subseteq \mathbb{Z}$, then

$$\left|\sum(A)\right| \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$$

Equality: $A = \{-m, \ldots, -1, 0, 1, \ldots, m\}$

Powerset sums

$A = \{a_1, \ldots, a_m\} \subseteq G$ abelian

$$\sum(A) := \left\{\sum B \mid B \subseteq A\right\}$$

$$= \{0, a_1\} + \{0, a_2\} + \ldots + \{0, a_m\}$$

$$\left|\sum(A)\right| \geq |H| \left(m + \text{\# of } a_i \text{ not in } H - (m-1)\right)$$

$$= |H| + |H| \cdot |A \setminus H|$$

Goal: $\left|\sum(A)\right| \geq |H| + c|A \setminus H|^2$

$$\left|\sum(A)\right| \geq 1 + \left\lfloor \frac{n^2}{4} \right\rfloor$$

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Toward $|\sum(A)| \geq |H| + c|A \setminus H|^2$

$$\left.\begin{array}{r} A \subseteq G \\ |A| \geq \sqrt{c\,p} \end{array}\right\} \Rightarrow \sum(A) = G$$

- (Erdös-Heilbronn 64) $G = \mathbb{Z}/p\mathbb{Z}$ $c = 54$
- (Olson 68) $c = 4$
- (Van Vu 06) $G = \mathbb{Z}/n\mathbb{Z}$ $A \subseteq U(G)$ c huge
  Uses (Szemeredi-Vu)

Theorem (DGMŠ, 08)

For $A \subseteq G$ Abelian, with $H = \mathrm{Stab}(\sum(A))$,

$$\left|\sum(A)\right| \geq |H| + \frac{1}{48}|A \setminus H|^2$$

Open: $\frac{1}{48} \mapsto \frac{1}{4} + \epsilon$

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Proof Outline

- Generalize to multisets A
- Reduce to case $H = \{0\}$
- (Inspiration: EH) $|A'| = m \Rightarrow \exists B \in \binom{A'}{\lfloor m/2 \rfloor}, |\sum(B)| \geq \frac{1}{64}m^2$
  - Grow B carefully.
  - Induct on m, keep track of $\langle A - B \rangle$.
  - Examine Cayley graphs in unbalanced cases
- $A =$
- (Kneser) $|\sum(B_i)| : \frac{m^2}{64} + \frac{m^2}{256} + \dots$
  $= \frac{m^2}{48} - O(\log m)$

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k-out-of-m sums

$\alpha = (a_1, a_2, \ldots, a_m)$ sequence from G

$\sum^k(\alpha) := \{a_{i_1} + a_{i_2} + \cdots + a_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq m\}$

Theorem (Erdös-Ginsberg-Ziv, 61)

$m \geq 2|G| - 1 \;\Rightarrow\; 0 \in \sum^{|G|}(\alpha)$

$\alpha = (\underbrace{0, \ldots, 0}_{n-1}, \underbrace{1, \ldots, 1}_{n-1})$ in $\mathbb{Z}/n\mathbb{Z}$

k-out-of-m sums

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$$\{a_1, a_p\} + \{a_2, a_{p+1}\} + \cdots + \{a_{p-1}, a_{2p-2}\} + \{a_{2p-1}\}$$

Either: $\exists i,\ a_i = a_{i+1} = \cdots = a_{i+p-1}$ (sum to 0)

or (CD) $|S| \geq \min\{p, (2p-1) - (p-1)\} = p$, so $0 \in S$.

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k-out-of-m sums

EGZ Generalizations

Theorem (Bollobás-Leader, 99 "shorter sequences")

$$|G| \leq m \leq 2|G| - 1 \;\Rightarrow\; \begin{cases} \left|\sum^{|G|}(\alpha)\right| \geq m - |G| + 1, \text{ or} \\ 0 \in \sum^{|G|}(\alpha) \end{cases}$$

Theorem (Hamidoune, 03 "t summands")

$$t \leq m \leq 2|t| - 1 \;\Rightarrow\; \begin{cases} \left|\sum^{t}(\alpha)\right| \geq m - t + 1, \text{ or} \\ \exists i,\ ta_i \in \sum^{t}(\alpha) \end{cases}$$

k-out-of-m sums

More EGZ Generalizations

Theorem (Grynkiewicz, 03 "many distinct summands')

$$\left.\begin{array}{r} m \geq |G| \\ \#\{\alpha\} \geq k \end{array}\right\} \Rightarrow \begin{cases} \left|\sum^{|G|}(\alpha)\right| \geq m - |G| + k - 1, \text{ or} \\ \exists H' \subseteq \sum^{|G|}(\alpha),\ \{0\} \neq H' \leq H \end{cases}$$

Conjecture (Gao, 96 "controlled repetitions")

$$\left.\begin{array}{r} G = \mathbb{Z}/n\mathbb{Z} \\ m \geq n + k - 1 \\ k \geq n/minprime(n) \end{array}\right\} \Rightarrow \begin{cases} 0 \in \sum^{|G|}(\alpha), \text{ or} \\ \mathrm{maxrep}(\alpha) \geq k \end{cases}$$

k-out-of-m sums

Cauchy-Davenport $\Rightarrow$ EGZ $G = \mathbb{Z}/p\mathbb{Z}$

Kneser $\Rightarrow$ EGZ G Abelian

??? $\Rightarrow$ EGZ Generalizations

k-out-of-m sums

Cauchy-Davenport $\Rightarrow$ EGZ $G = \mathbb{Z}/p\mathbb{Z}$

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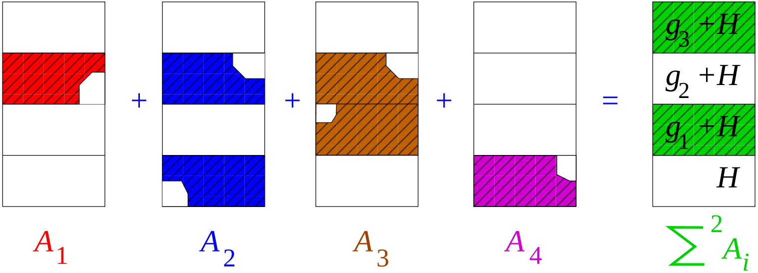
"k-out-of-m Knesser" $\Rightarrow$ EGZ Generalizations

k-out-of-m Kneser

$$A = (A_1, A_2, \ldots, A_m), \quad A_i \in G$$

$$\sum^k(A) := \{A_{i_1} + A_{i_2} + \cdots + A_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le m\}$$



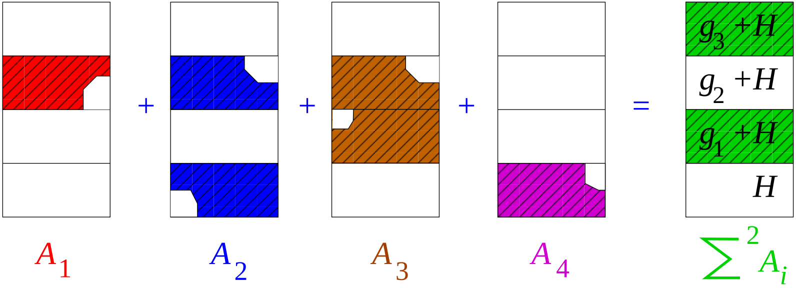
k-out-of-m Kneser

$$A = (A_1, A_2, \ldots, A_m),\ \ A_i \in G \qquad H := \mathrm{Stab}\left(\sum^k(A)\right)$$

Theorem (DGM, 07)

$$\left|\sum^k(A)\right| \geq |H|\ (\textit{total \# represented } H\textit{-cosets}^* - \textit{\# additions})$$

* *no coset may be counted* $> k$ *times.*



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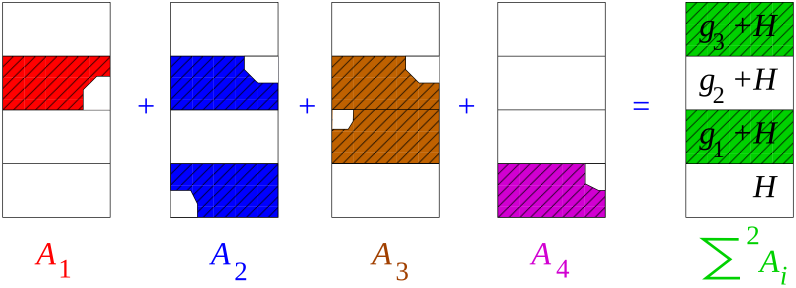
$$A = (A_1, A_2, \ldots, A_m),\ \ A_i \in G \qquad H := \text{Stab}\left(\sum\nolimits^k (A)\right)$$

Theorem (DGM, 07)

$$\left|\sum\nolimits^k A_i\right| \geq |H| \left(\sum\nolimits_{Q \in G/H} \min(k, \{i \mid A_i \cap Q \neq \emptyset\}) - (k-1)\right)$$



k-out-of-m Kneser

Proof Features

- Generalize: K-coset patterns, arbitrary $K \leq G$
- $\min \left|\sum^k(A)\right|$; $\min \sum_{i=1}^m |A_i|$; $\max \sum_{i=1}^m |A_i|^2$; $\min m$
- WMA $H = \{0\}$
- Special case: $k = m$ and $\exists i \ |A_i + K| > |K|$
- Set up $A' = (A_1 \cap A_2, A_1 \cup A_2, A_3, \ldots, A_m)$
- Convergents and their stabilizers:

  Sums: $\sum^k(A') \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq \sum^k(A)$

  Stabs: $H' \geq H_1 \geq H_2 \geq \cdots \geq \{0\}$
- $\exists t,\ C_t = \sum^k(A)$

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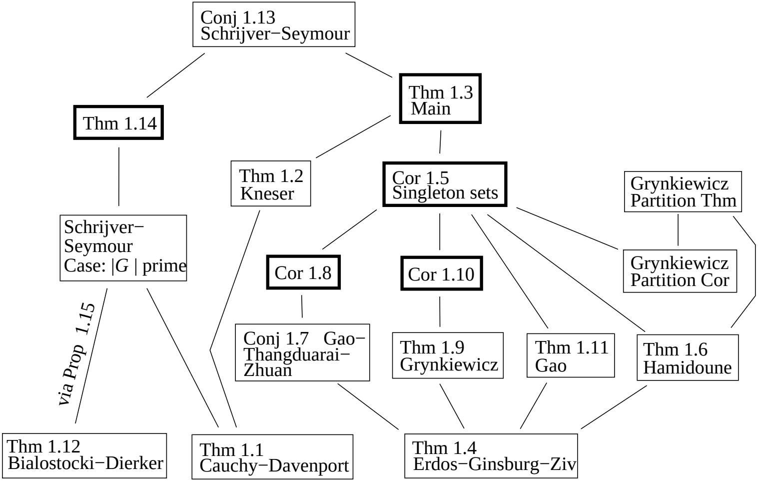
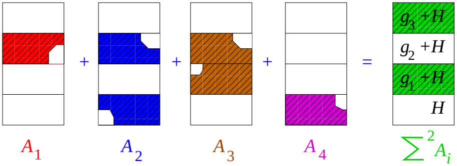
Dependence among results



Vector formulation of “k-out-of-m”

$E = G \times [m], \quad e = (g, i)$
G-weight: $w(e) := g$



$v_1, \ldots, v_m \in \mathbb{R}^k$
in general position

$$v(e) := \begin{cases} v_i & \text{if } w(e) \in A_i \\ \mathbf{0} & \text{if } w(e) \notin A_i \end{cases}$$

Then $\sum^k(A) = \{w(B) \mid v(B) \text{ is a basis of } \mathbb{R}^k\}$

$w(B) := \sum\{w(b) \mid b \in B\}$
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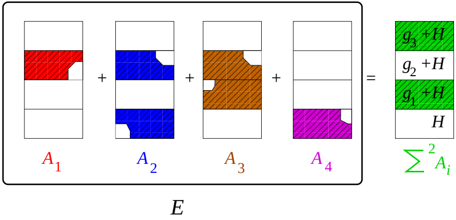

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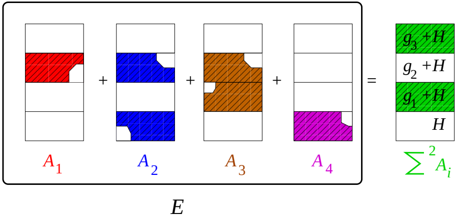

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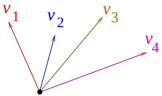

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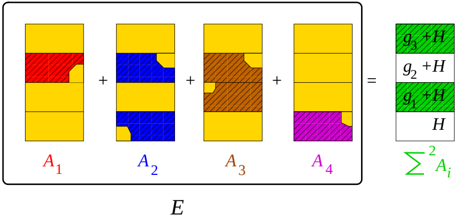

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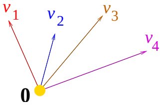

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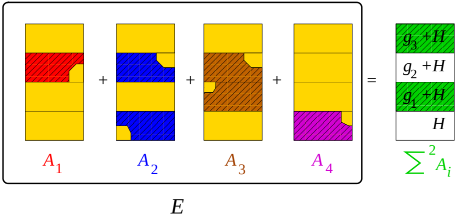

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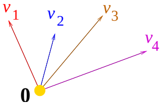

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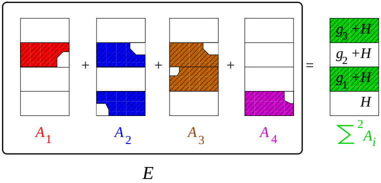
Vector formulation of “k-out-of-m”

DGM Theorem Restated:

$w : E \to G$

$\mathcal{B} := \{B \subseteq E \mid v(B) \text{ is basis } \}$

$w(\mathcal{B}) = \{w(B) \mid B \in \mathcal{B}\}$



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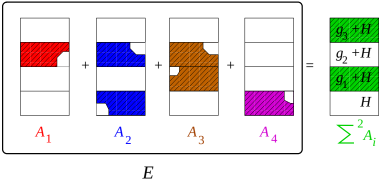

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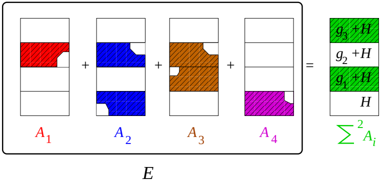

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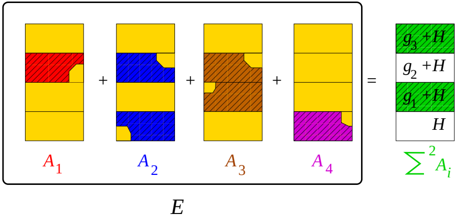
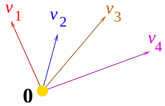
$$|w(\mathcal{B})| \geq |H| \left(\sum_{Q \in G/H} \text{rk}(w^{-1}(Q)) - (k-1)\right)$$

Vector-based generalization of “2 -out-of-3”

Variations:



eg. Lift v_1 out of plane.



$w(\mathcal{B}) = A_1+$ “add any 2 out of A_3, A_3, A_4”

$|w(\mathcal{B})| \geq$???

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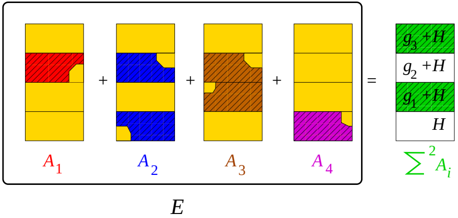

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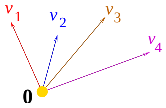

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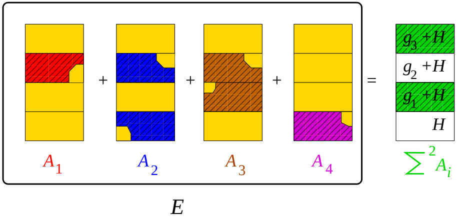

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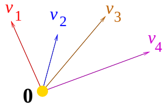

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$|w(\mathcal{B})| \geq |H| \left(\sum_{Q \in G/H} \mathrm{rk}(w^{-1}(Q)) \; - \; (k-1) \right)$ (Conjecture?)

Matroid-based generalization of “2 -out-of-3”

Conjecture (Seymour-Schrijver, 91)

For:

- G *Abelian*
- $\mathcal{B} \subseteq 2^E$ *a set of bases of a vector space of rank* k
- $w : E \to G$,

we have

$$|w(\mathcal{B})| \geq |H| \left(\sum_{Q \in G/H} \mathrm{rk}(w^{-1}(Q)) \;-\; (k-1) \right)$$

“Bases of Many Weights”

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Conjecture (Seymour-Schrijver, 91)

For:

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“Bases of Many Weights”

Group-weighted matroid conjecture

Conjecture (Seymour-Schrijver, 91)

For any Abelian-group weighted matroid $w : M \to G$,

$$|w(\mathcal{B}(M))| \geq |H| \left(1 - \mathrm{rk}(M) + \sum_{Q \in G/H} \mathrm{rk}\left(w^{-1}(Q) \right) \right)$$

True for:

- Rank 2 matroids (Kneser)
- Uniform matroids (EGZ)
- Uniform + parallel matroids (DGM)
- $|G| =$ prime (Seymour-Schrijver)
- $|G| =$ prime power (DGM)
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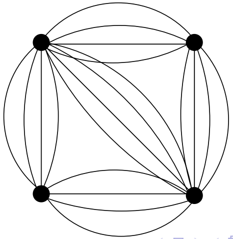
Unknown:

- K_4^-: Column vectors:

1	0	0	1	0
0	1	0	1	1
0	0	1	0	1



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EGZ for Group-weighted matroids

Vanishing Basis Question:

$$\left.\begin{array}{c} |G| = \mathrm{rk}(M) \\ w : M \to G \end{array}\right\} \Rightarrow 0 \in w(\mathcal{B}(M)) \text{ ?????}$$

Yes for:

- Uniform matroids of size $> 2|G| - 1$ (Erdös-Ginsberg-Ziv)
- $|G| = p^k$ or pq and $M(K_{p+1})$ (Bialostokie-Dierker, DGM)

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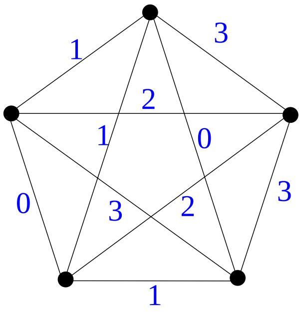
eg. Spanning trees of K_5



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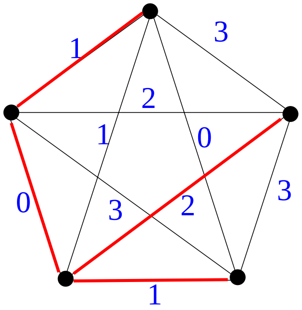
eg. Spanning trees of K_5



Spanning tree of weight 0 mod 4

EGZ for Group-weighted matroids

Vanishing Basis Question:

$$\left.\begin{array}{c}|G| = \mathrm{rk}(M) \\ w : M \to G\end{array}\right\} \Rightarrow 0 \in w(\mathcal{B}(M)) \text{ ?????}$$

Yes for:

- Uniform matroids of size $> 2|G| - 1$ (Erdös-Ginsberg-Ziv)
- $|G| = p^k$ or pq and $M(K_{p+1})$ (Bialostokie-Dierker, DGM)
- Certain dual paving matroids, where G is cyclic (DGM)
- If any two cocircuits of M meet, and SS holds for (M^*, G) (Seymour-Schriver, DGM)

EGZ for Group-weighted matroids

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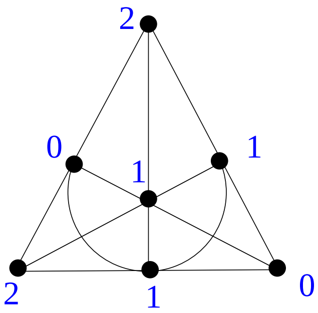
eg. Projective Geometries



EGZ for Group-weighted matroids

Vanishing Basis Question:

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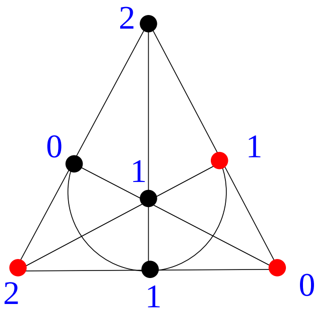
eg. Projective Geometries



Noncollinear triple of weight 0 mod 3

Happy Birthday!!!