Generalized Vandermonde Determinants, Interpolation and Splines

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Ideas coming out of teaching linear algebra and geometric modeling to game programmers

Outline of Talk

1. Vandermode-type determinants
    - Lagrange Interpolation (Vandermonde det)
    - Hermite Interpolation (Confluent det)
    - Birkhoff Interpolation (? det)
2. Polyonmial splines
    - Recursive Algorithms and Geometry
    - Standard Basis
    - B-spline Basis
3. Simple basis change (Theorem of Curry and Schoenberg)

Examples:

Vandermonde Determinant:

$$\begin{vmatrix} 1 & a_0 & a_0^2 & a_0^3 & a_0^4 \\ 1 & a_1 & a_1^2 & a_1^3 & a_1^4 \\ 1 & a_2 & a_2^2 & a_2^3 & a_2^4 \\ 1 & a_3 & a_3^2 & a_3^3 & a_3^4 \\ 1 & a_4 & a_4^2 & a_4^3 & a_4^4 \end{vmatrix} = D(a_1 a_2 a_3 a_4 a_5)$$

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Confluent Vandermonde Determinant:

$$\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ 0 & 1 & 2a & 3a^2 & 4a^3 \\ 0 & 0 & 2 & 6a & 12a^2 \\ 1 & b & b^2 & b^3 & b^4 \\ 0 & 1 & 2b & 3b^2 & 4b^3 \end{vmatrix} = D(aaabb)$$

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“Birkhoff” Determinant:

$$\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ 0 & 1 & 2a & 3a^2 & 4a^3 \\ 0 & 0 & 0 & 6 & 24a \\ 1 & b & b^2 & b^3 & b^4 \\ 0 & 1 & 2b & 3b^2 & 4b^3 \end{vmatrix} = D(aa * abb)$$

Product Formulas:

Vandermonde Determinant:

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Connections to Interpolation

The Vandermonde Matrix:

$$\begin{pmatrix} 1 & a_0 & a_0^2 & a_0^3 & a_0^4 \\ 1 & a_1 & a_1^2 & a_1^3 & a_1^4 \\ 1 & a_2 & a_2^2 & a_2^3 & a_2^4 \\ 1 & a_3 & a_3^2 & a_3^3 & a_3^4 \\ 1 & a_4 & a_4^2 & a_4^3 & a_4^4 \end{pmatrix}$$

is the coefficient matrix of the linear system for the Lagrange interpolation problem:

Find $p(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$ such that $p(a_i) = y_i$ for some chosen values y_i.

Using Cramer's Rule, we can solve for the A_i. In particular,

$$A_4 = \frac{1}{D} \sum_{i=0}^{4} (-1)^i y_i D(s_i)$$

where s_i is the string $a_1 a_2 a_3 a_4 a_5$ with a_i omitted.

(Aside: Newton Form)

In the language of divided differences, if the values $y_i = g(a_i)$ for some function g, then:

$$A_4 = [a_0 a_1 a_2 a_3 a_4]g = \frac{1}{D}\sum_{i=0}^{4}(-1)^i y_i D(s_i).$$

We can then find $p(x)$ in the Newton Form using the recursion for divided differences:

$$p(x) = [a_0]g + ([a_0a_1]g)(x-a_0) + ([a_0a_1a_2]g)(x-x_0)(x-x_1)+$$
$$\cdots + ([a_0a_1a_2a_3a_4]g)(x-a_0)(x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

where the coefficients satisfy the divided difference recursion:

$$[a_i, a_{i+1}, \ldots, a_{i+k}]g$$

$$= \frac{[a_{i+1}, \ldots, a_{i+k}]g - [a_i, \ldots, a_{i+k-1}]g}{a_{i+k} - a_i}.$$

The Confluent Vandermonde Matrix:

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is the coefficient matrix of the linear system for the Hermite interpolation problem:

Find $p(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$ such that $p(a) = y_0$, $p'(a) = y_1$, $p''(a) = y_2$, $p(b) = y_3$, $p'(b) = y_4$, for some chosen values y_i.

This Hermite polynomial can also be found in the Newton form, where now the divided difference is modified in the case of k equal values:

$$[aa \cdots a]g = g^{(k)}(a)/k!.$$

The Birkhoff Matrix:

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is the coefficient matrix of the linear system for the Birkhoff interpolation problem:

Find $p(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4$ such that $p(a) = y_0$, $p'(a) = y_1$, $p'''(a) = y_2$, $p(b) = y_3$, $p'(b) = y_4$, for some chosen values y_i.

Open Problem: In the general case, the Birkhoff interpolation problem may or may not have a unique solution. No general form for the determinant is known, and thus no general formula for the interpolating polynomial (when it exists) is known.

In particular, is there a compact product formula for all $D(s)$ where s contains only one star, ie. is missing just one derivative?

Some Product Formulas:

Vandermonde Product Formula:

$$\begin{vmatrix} 1 & a_0 & a_0^2 & a_0^3 & \dots & a_0^d \\ 1 & a_1 & a_1^2 & a_1^3 & \dots & a_1^d \\ 1 & a_2 & a_2^2 & a_2^3 & \dots & a_2^d \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_d & a_d^2 & a_d^3 & \dots & a_d^d \end{vmatrix} = \prod_{i<j} (a_j - a_i)$$

String Notation:

$$D(a_0 a_1 a_2 \dots a_d) = \prod_{i<j} (a_j - a_i)$$

Confluent Vandermode Product Formula

Let $s = a_1^{e_1} \cdots a_n^{e_n}$, with the a_i distinct.

Then $D(s) = \prod_{i<j} (a_j - a_i)^{e_i e_j} \prod_{k=1}^{n} (e_k - 1)!!$.

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Sample Birkhoff Product Formula

Let $s = a^{k-1} * a b_1^{e_1} \cdots b_r^{e_r}$,

with $a \neq b_i \neq b_j$, for $i \neq j$.

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Let $s = a^{k-1} * ab_1^{e_1} \cdots b_r^{e_r}$,

with $a \neq b_i \neq b_j$, for $i \neq j$.

Then

$$\begin{aligned} D(s) &= k!(k-2)!! \prod_{i=1}^{r} (e_i - 1)!! \\ &* \prod_{i=1}^{r} (b_i - a)^{(k-1)e_i} \prod_{i<j} (b_j - b_i)^{e_i e_j} \\ &* \left[\sum_{i=1}^{r} e_i (a - b_i)^{e_i - 1} \prod_{j \neq i} (a - b_j)^{e_j} \right] \end{aligned}$$

Proof in the Confluent Case

$$D(a_1^{e_1} \cdots a_n^{e_n}) = \prod_{i<j} (a_j - a_i)^{e_i e_j} \prod_{k=1}^{n} (e_k - 1)!!.$$

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Let $e_m = K(s)$ for some index m, and let r be the string which is indentical to s except that the last occurence of a_m is replaced by x. So r looks like:

$$r = a_1^{e_1} \cdots a_m^{e_m - 1} \; x \; a_{m+1}^{e_{m+1}} \cdots a_n^{e_n}$$

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Then $D(r) = p(x)$ is a polynomial in x of degree at most d.

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Then $D(r) = p(x)$ is a polynomial in x of degree at most d.

Note also that $J(r) = J(s) - 1$. (for induction)

The derivatives of $p(x)$ can be obtained by differentiating the row of the matrix containing x, and taking the new determinant. By repeated differentiation, we can see that $D(s)$ is simply a derivative of p evaluated at a_m:

$$D(s) = p^{(e_m - 1)}(a_m).$$

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Example: $s = aaab$

$$D(s) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 2 & 6 * a \\ 1 & b & b^2 & b^3 \end{vmatrix}.$$

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$$P(x) = D(aaxb) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 1 & x & x^2 & x^3 \\ 1 & b & b^2 & b^3 \end{vmatrix}.$$

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$$P''(x) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 2 & 6x \\ 1 & b & b^2 & b^3 \end{vmatrix}, \text{ and } P''(a) = D(aaab).$$

Now since

$$r = a_1^{e_1} \cdots a_m^{e_m - 1} \; x \; a_{m+1}^{e_{m+1}} \cdots a_n^{e_n},$$

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$$p(x) \;\; = \;\; (e_m - 2)!! \prod_{k \neq m} (e_k - 1)!! \prod_{i<j} (a_j - a_i)^{e'_i e'_j}$$

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(where $e'_m = e_m - 1$ and $e'_i = e_i$, $i \neq m$.)

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where $q_1(x) = \prod_{l<m} (x - a_l)^{e_l} \prod_{m<p} (a_p - x)^{e_p}$, and thus $q_1(a_m) \neq 0$.

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Differentiating:

$$p^{(e_m - 1)}(x) = (constant)[(e_m - 1)!q_1(x) + q_2(x)]$$

with $q_2(a_m) = 0$.

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So: $q_1(x) = \prod_{l<m} (x - a_l)^{e_l} \prod_{m<p} (a_p - x)^{e_p}$,

and

$$\begin{aligned} D(s) &= p^{(e_m - 1)}(a_m) = (constant)(e_m - 1)!q_1(a_m) \\ &= \left((e_m - 2)!! \prod_{k \neq m} (e_k - 1)!! \prod_{i<j} (a_j - a_i)^{e_i' e_j'} \right) \\ &* \ (e_m - 1)!q_1(a_m) \end{aligned}$$

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&= \prod_{i<j} (a_j-a_i)^{e_i e_j} \prod_{k=1}^{n} (e_k-1)!!
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$\square$.

Connections with polynomial splines

Let V be the vector space of piecewise polynomials $f(x)$ on the sequence of intervals $[u_0, u_1, \dots u_k]$ such that on each subinterval f restricts to a polynomial of degree at most d, and f is C^{r_i} (has r_i continuous derivatives) at u_i, for $i = 1, \dots, k-1$.

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Then V has the standard (truncated power) basis:

$$\{1, x, x^2, \dots, x^d, (x-u_1)_+^{r_1+1}, \dots, (x-u_1)_+^d,$$

$$(x-u_2)_+^{r_2+1}, \dots, (x-u_2)_+^d$$

$$\dots$$

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Note: The above basis has a "left-handed" counterpart given by functions of the form $(u_i - x)_+^k$ (replacing the corresponding right-handed functions.)

A basic fact (proved in 1947 by Curry and Schoenberg) is that V always has a basis made up of certain functions with “minimal support” called B-splines.

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In order to define such a basis, we extend the sequence $u_0, \ldots, u_k$ first to include multiplicities $m_i = d - r_i$ for each of the u_i, and then to:

$$t_0, \ldots, t_d = u_0, \ldots, t_{N-d} = u_k, \ldots, t_N.$$

This “knot sequence” then encodes the orders of continuity at $u_1, \ldots, u_{k-1}$ with the multiplicities.

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For $t_0 \le t_1 \le \ldots \le t_N$, and for $0 \le i \le N - d - 1$,

$$\mathcal{B}_i^d(t) = (t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](x - t)_+^d$$

Recursion and Stable Evaluation

Many books on splines are now written by computer graphics specialists and typically give the recursive formula (DeBoor and Cox - 1972) as the starting point for B-splines:

$$\mathcal{B}_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i}\mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}}\mathcal{B}_{i+1}^{d-1}(t).$$

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The above leads to algorithms for stable evaluation of spline functions and to nice geometric algorithms for evaluation of spline curves and surfaces.

For example, a B-spline curve can be evaluated using Nested Linear Interpolation, in a similar way as the polynomial Bezier curve can be evaluated:

$$\gamma(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + (1-t) t^2 P_2 + t^3 P_3$$

Curry-Schoenberg

Let \$ be the vector space spanned by the B-splines, $\mathcal{B}_i^d(t)$, $i = 0, \ldots, N-d-1$, restricted to the interval $[t_d, t_{N-d}]$.

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The basic idea is surprisingly simple: We write the divided difference in the definition of $\mathcal{B}_i^d(t)$ as a quotient of determinants via Cramer's Rule.

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The denominator is a confluent Vandermonde determinant, and the numerator can be expanded as a sum of "left-handed" truncated power basis functions with coefficients which are "almost" confluent Vandermonde. (ie. these are $D(s)$ with s having at most one star.)

The matrix of column (coordinate) vectors can then be seen to be lower triangular, with nonzero diagonal entries, thus giving both the proof that the B-splines are a basis, and the change of basis matrix.

Example:

Applying this to the B-spline definition:

$$\mathcal{B}_i^d(t) = (t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](x-t)_+^d$$

for the sequence a, b, c:

$$\mathcal{B}_0^1(t) = (c-a)[a,b,c](x-t)_+^2$$

$$= \frac{1}{D(abc)} \begin{vmatrix} 1 & a & (a-t)_+^2 \\ 1 & b & (b-t)_+^2 \\ 1 & c & (c-t)_+^2 \end{vmatrix}$$

$$= \frac{D(bc)}{D(abc)}(a-t)_+^2 - \frac{D(ac)}{D(abc)}(b-t)_+^2 + \frac{D(ab)}{D(abc)}(c-t)_+^2$$