Peter Borwein, Plane Geometry, Polynomials, and Polygons

$$x^{54} - x^{53} + x^{52} - x^{51} + x^{44} - x^{43} + x^{42} - x^{41} + x^{40} - x^{33} + x^{32} - x^{31} + x^{30} - x^{29} + x^{22} - x^{21} + x^{20} - x^{19} + x^{18} - x^{11} + x^{10} - x^{9} + x^{8} - x^{7} + 1$$

Michael Mossinghoff
Davidson College

Outline

I. Peter’s interests in **plane geometry**.

II. Peter’s interests in **polynomials** with restricted coefficients and prescribed vanishing.

III. A problem that **combines** these interests!

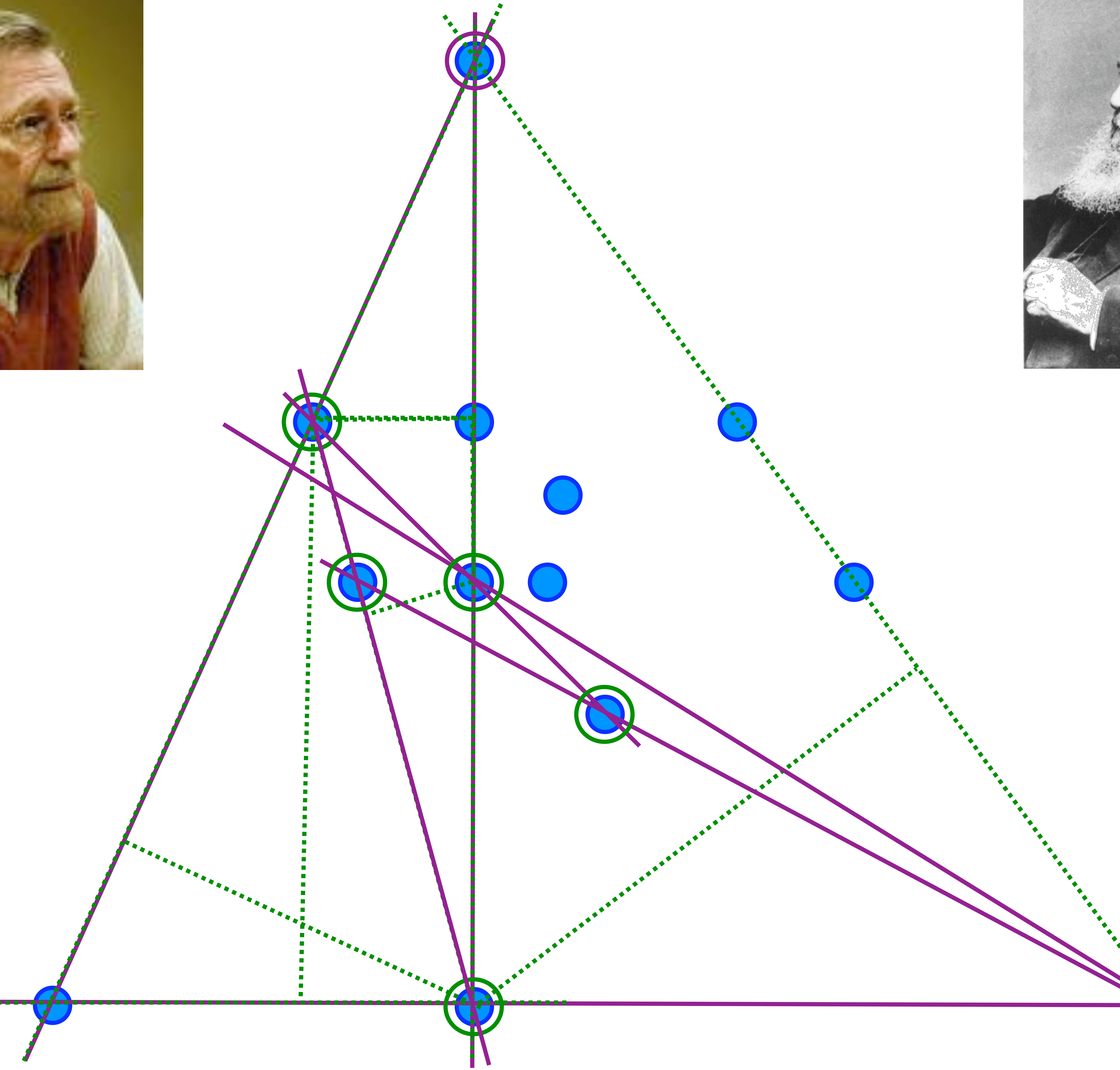
I.
PLANE GEOMETRY

Sylvester's Problem

J. J. Sylvester (1893): Suppose $n \geq 3$ points in the plane do not all lie on the same line. Does there exist a line that passes through exactly two of the points?

L. M. Kelly (c. 1948): "Book proof."

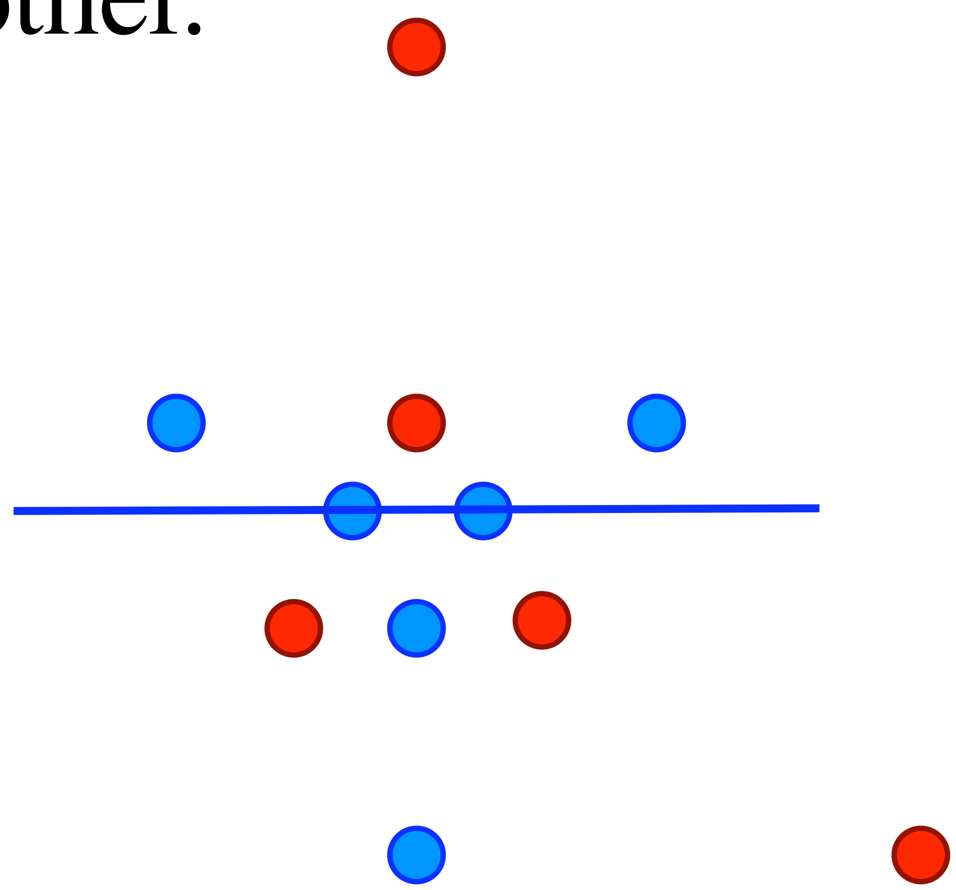
E. W. Dijkstra (1988): Algorithmic solution.


Motzkin's Theorem

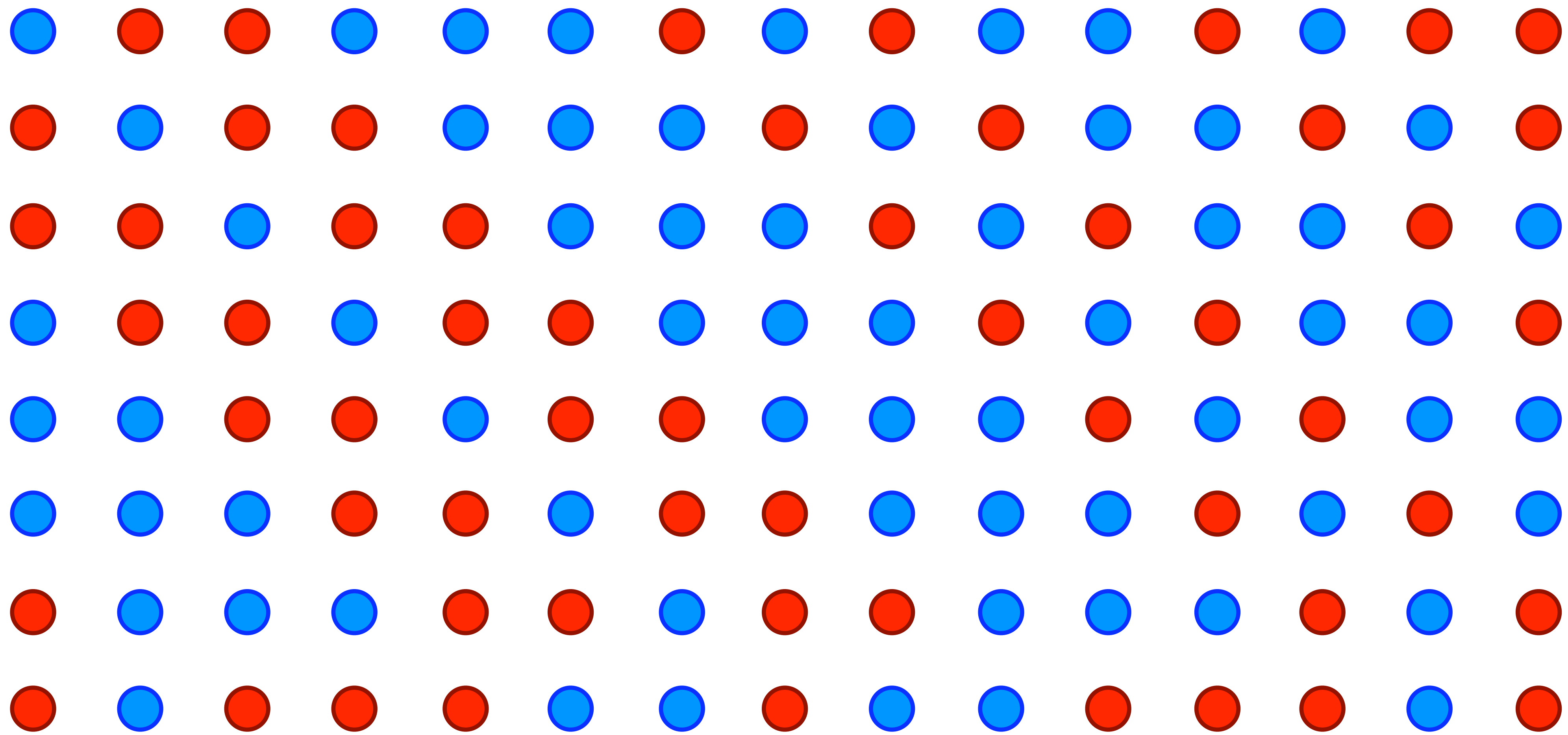
T. S. Motzkin, 1967: If A and B are finite, disjoint sets of points in the plane, and $A \cup B$ does not lie on a line,

then there exists a line passing through at least two points of one set and no points of the other.



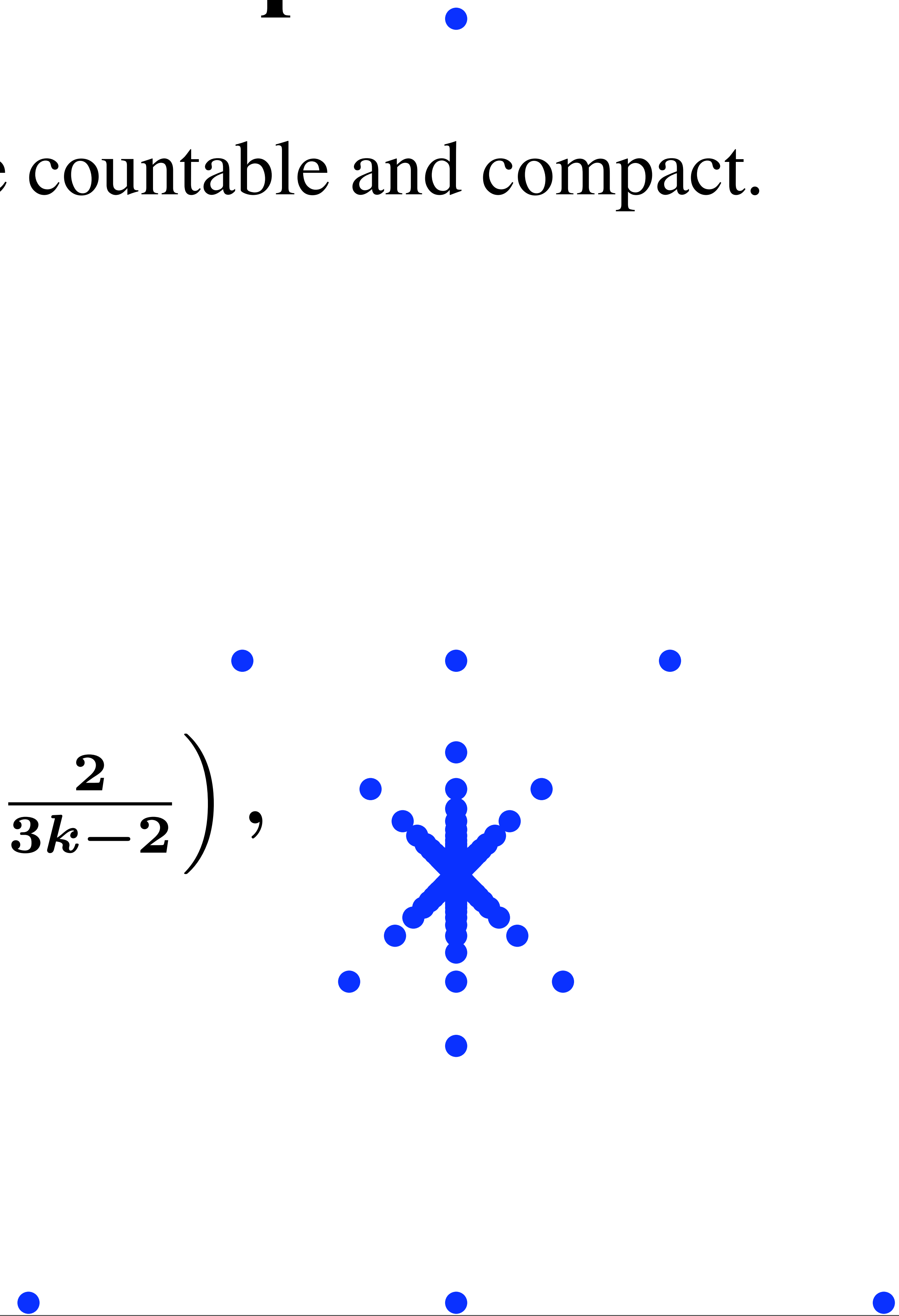
Countably Many Points?

- Sylvester: No!
- Motzkin: No!



Countably Many & Compact?

- Suppose A (and B) are countable and compact.
- Sylvester? Motzkin?
- Sylvester: No!
- $\left(\frac{\pm 1}{3k-1}, \frac{1}{3k-1}\right), \ \left(0, \frac{2}{3k-2}\right),$ $(0,0)$.



Peter's Results

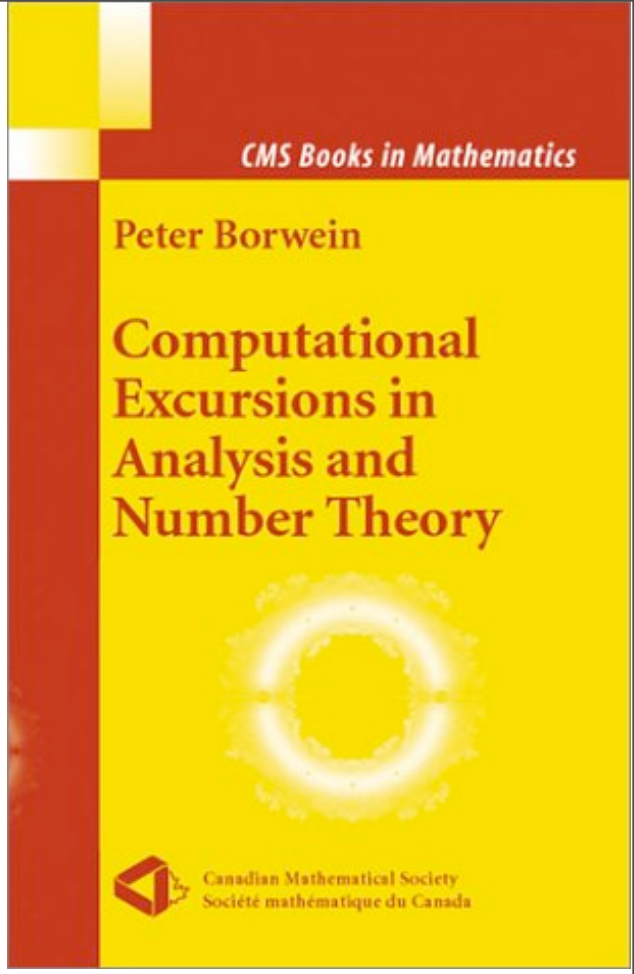
- Proc. Amer. Math. Soc., 1984.
- Motzkin's Theorem for infinite sets:
  - **True** if A and B are compact, countable, and disjoint, unless A and B are contained in a single line.
- Sylvester's Theorem for infinite sets:
  - If A is non-collinear, countable, and compact, then there exists a line that passes through only **finitely many** points of A.

More Results

- Pacific J. Math, 1983.
- Sylvester's Problem for higher degree interpolants:
    - Suppose A is a finite set of points in the plane with distinct x-coordinates ($|A| > n+1$).
    - Suppose A does not lie on a polynomial curve of degree $\leq n$.
    - Then there exists a polynomial of degree n passing through exactly $n + 1$ points of A.

II.
Polynomials

Common Thread



Many of Peter's research projects concern polynomials with:

- Restricted coefficients.
    - Unimodular, {-1, 0, 1}, {0, 1}.
- Requirement on vanishing.
    - Order at fixed point, roots on unit circle, etc.
- Optimality condition.
    - Degree, number of terms, norm.

Example: {-1, 0, 1} Coefficients

1. Find a polynomial with $\{-1, 0, 1\}$ coefficients with a zero of order $m \geq 5$ at z with $|z| \notin \{0, 1\}$.

- Mahler's measure of monic $f(x)$:

$$M(f) = \prod_{f(\beta)=0} \max\{1, |\beta|\}.$$

- Lehmer's Question (1933):
If $f(x)$ is monic, irreducible, noncyclotomic, is $M(f) > 1 + \epsilon$ for some absolute $\epsilon > 0$?

- Pathiaux (1973): If $M(g) < 2$ then there exists a polynomial G with $\{-1, 0, 1\}$ coefficients and $g \mid G$.

- So if $1 < M(f) < 2^{1/m}$, then f^m divides a $\{-1, 0, 1\}$ polynomial.

- Or, if the multiplicity of a noncyclotomic factor of a $\{-1, 0, 1\}$ polynomial is bounded, then Lehmer's question is resolved.

- Lehmer:

  $$\ell(x) = x^{10}+x^{9}-x^{7}-x^{6}-x^{5}-x^{4}-x^{3}+x+1$$

  has $M(\ell) = 1.1762808\ldots < 2^{1/4}$.

- Thus polynomial with $m = 4$ exists for $\ell(x)$.

- Best constructed: $m = 3$:

$$\ell(x)^{3}\ell(-x)\Phi_1^2\Phi_2\Phi_3\Phi_4^2\Phi_6\Phi_7\Phi_{11}\Phi_{12}\Phi_{18}\Phi_{30}\Phi_{32}\Phi_{48} =$$

+0−0−+000−+−+00−00−0000+−00+−+−−00−0++0+−−++−00−0+0+++−+−−+
+−−+−+++0+0−00−++−−+0++0−00−−+−+00−+0000−00−00+−+−000+−0−0+

- Bombieri and Vaaler (1983):
  For $\ell^4(x)$, degree $< \mathbf{16000}$.

- Is there any example with $m \geq \mathbf{5}$?

- Beaucoup, Borwein, Boyd, Pinner (J. London Math. Soc., 1998):

  For $\ell(x)$, maximum multiplicity is at most $\mathbf{6}$.

Example: {-1, 0, 1} Coefficients

2. Find a polynomial with $\{-1, 0, 1\}$ coefficients with a zero of order m at $z = 1$ having just $2m$ terms.

$\longleftrightarrow$ Prouhet-Tarry-Escott problem (c. 1910):

- Find A, B, disjoint sets of m integers with

$$\sum_{a \in A} a^k = \sum_{b \in B} b^k$$

for $0 \leq k < m$.

- Example ($\boldsymbol{m = 5}$):

$$\boldsymbol{A = \{19, 15, 11, 3, 2\}},$$

$$\boldsymbol{B = \{18, 17, 9, 5, 1\}}.$$

$$\Big\updownarrow$$

$$\boldsymbol{f(x) = x^{19} - x^{18} - x^{17} + x^{15} + x^{11} - x^{9} - x^{5} + x^{3} + x^{2} - x.}$$

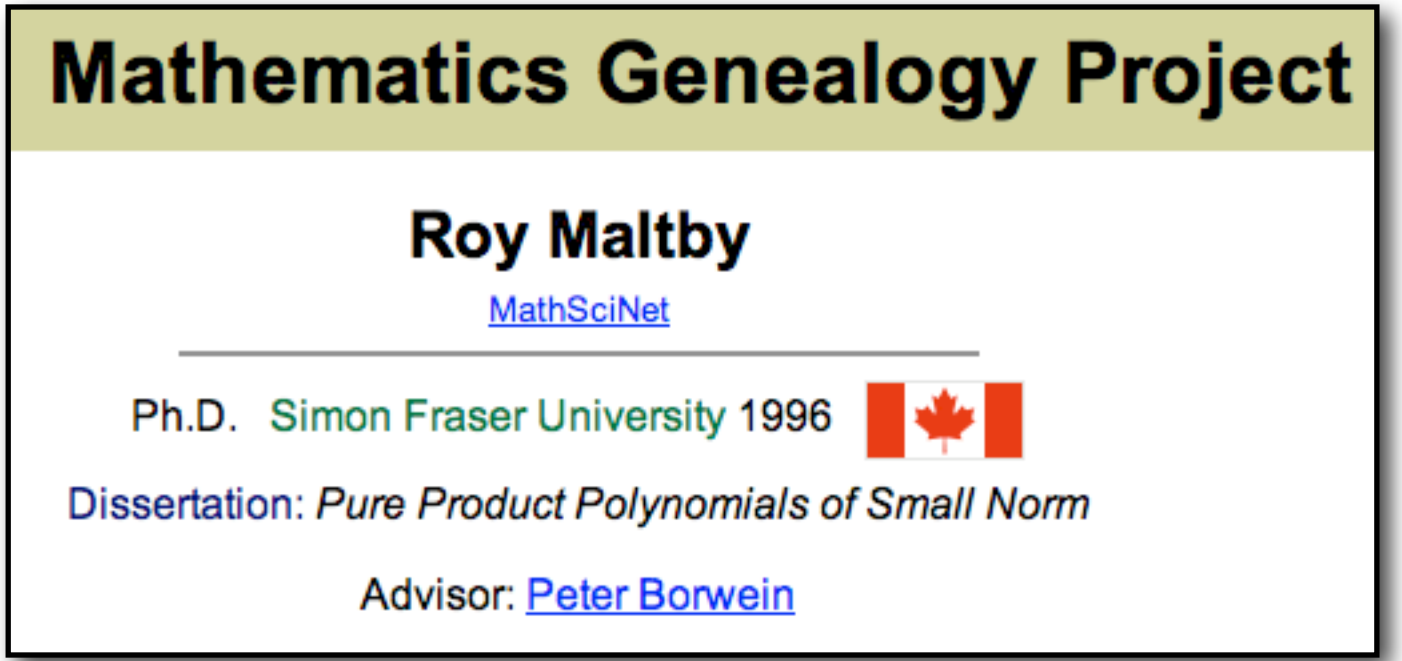
- $\boldsymbol{(x-1)^5 \mid f(x)}$, ten terms.

- Borwein and Ingalls (Enseign. Math., 1994).

- Borwein, Lisonek, Percival (Math. Comp., 2003).

  - New small solutions for $n = \mathbf{10}$:

$$\boldsymbol{A = \{\pm 71, \pm 131, \pm 308, \pm 180, \pm 307\},}$$
$$\boldsymbol{B = \{\pm 99, \pm 100, \pm 301, \pm 188, \pm 313\}.}$$

- 

Mathematics Genealogy Project

Roy Maltby

MathSciNet

Ph.D. Simon Fraser University 1996

Dissertation: *Pure Product Polynomials of Small Norm*

Advisor: Peter Borwein

III. Polygons and Polynomials

A Problem Peter May Like . . .

Problem in plane geometry involving arrangements of points and lines.

↕

Problem about polynomials with:

- {-1, 0, 1} coefficients.
- Prescribed vanishing.
- Restrictions on number of terms and coefficient patterns.

Isoperimetric Problems

- Fix perimeter; maximize area.
- General planar curves: circle is optimal.
- Polygon with fixed number of sides, ***n***:
  - Regular ***n***-gon is optimal.
  - Unique solution.
- $A \leq L^2/4n \cot(\pi/n)$.

Isodiametric Problems

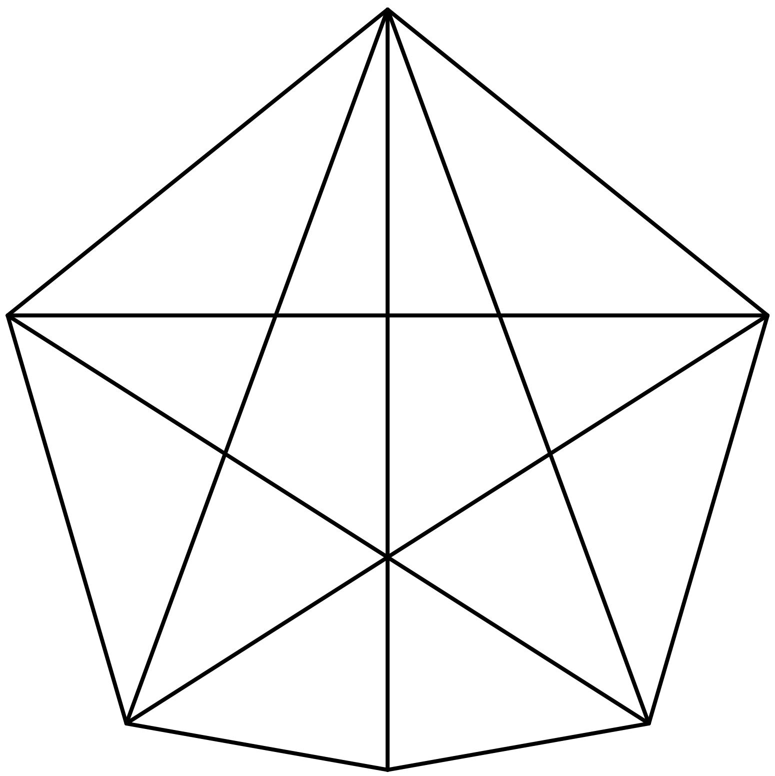
- **Problem 1:** Fix diameter; maximize area.
- **Problem 2:** Fix diameter; maximize perimeter (for a convex set).
- General planar curves: circle is optimal.
  - Area: Bieberbach, 1915.
  - Perimeter: Rosenthal and Szász, 1916.
- Polygons: Reinhardt, 1922.

Reinhardt's Results: Area Problem

- ***n* odd:**
  - Regular n-gon is optimal.
  - Unique solution.
- ***n* even:**
  - Square is optimal, though not uniquely.
  - Regular n-gon never optimal for $n \geq 6$.
  - Left open: optimal n-gon for even n?

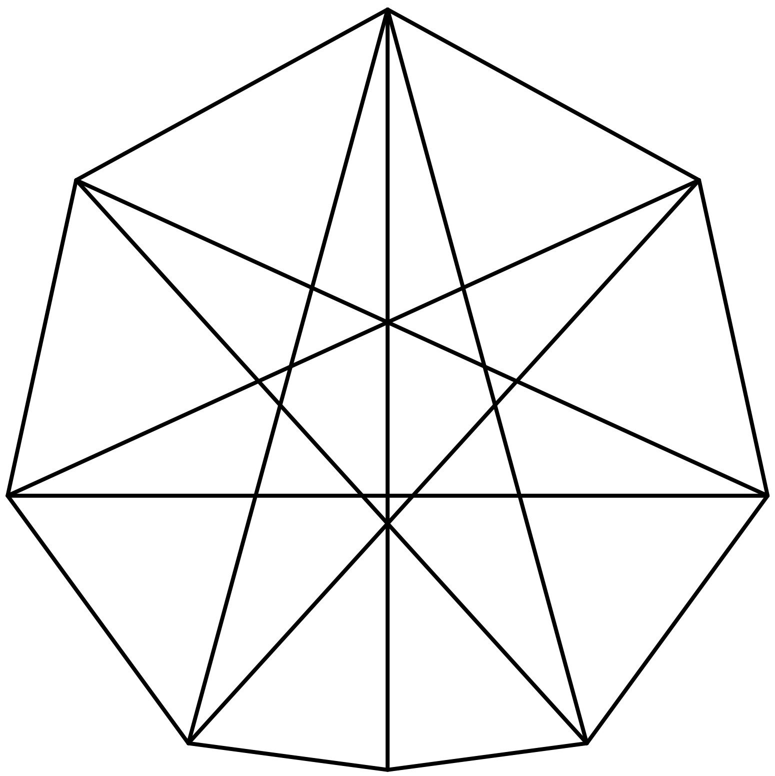
Later Work: Area Problem

Optimal Hexagon



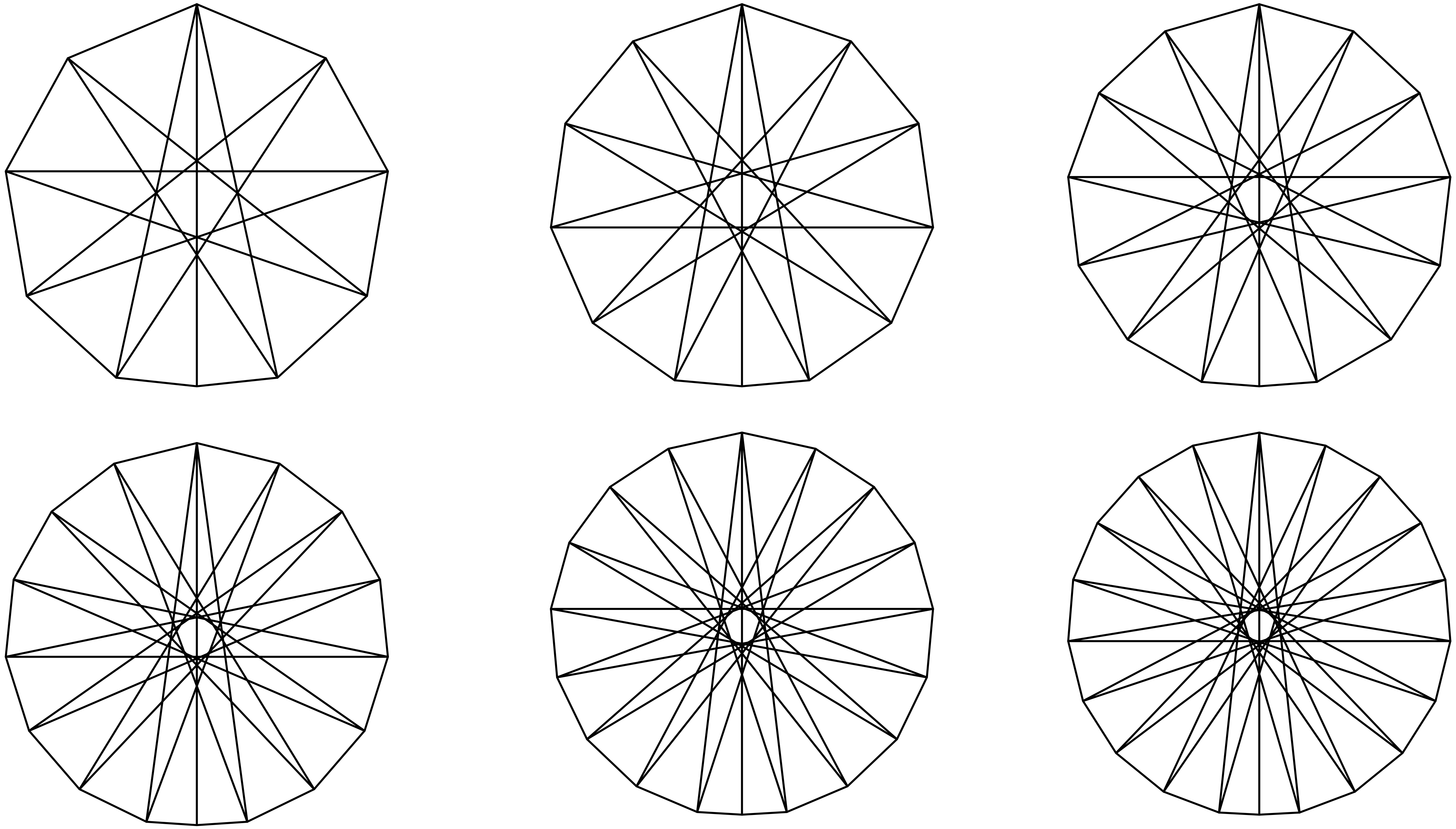
Graham, 1975.

Optimal Octagon



Audet, Hansen, Messine, Xiong, 2002.

Quantitative Improvement for all even *n*



M., 2006.

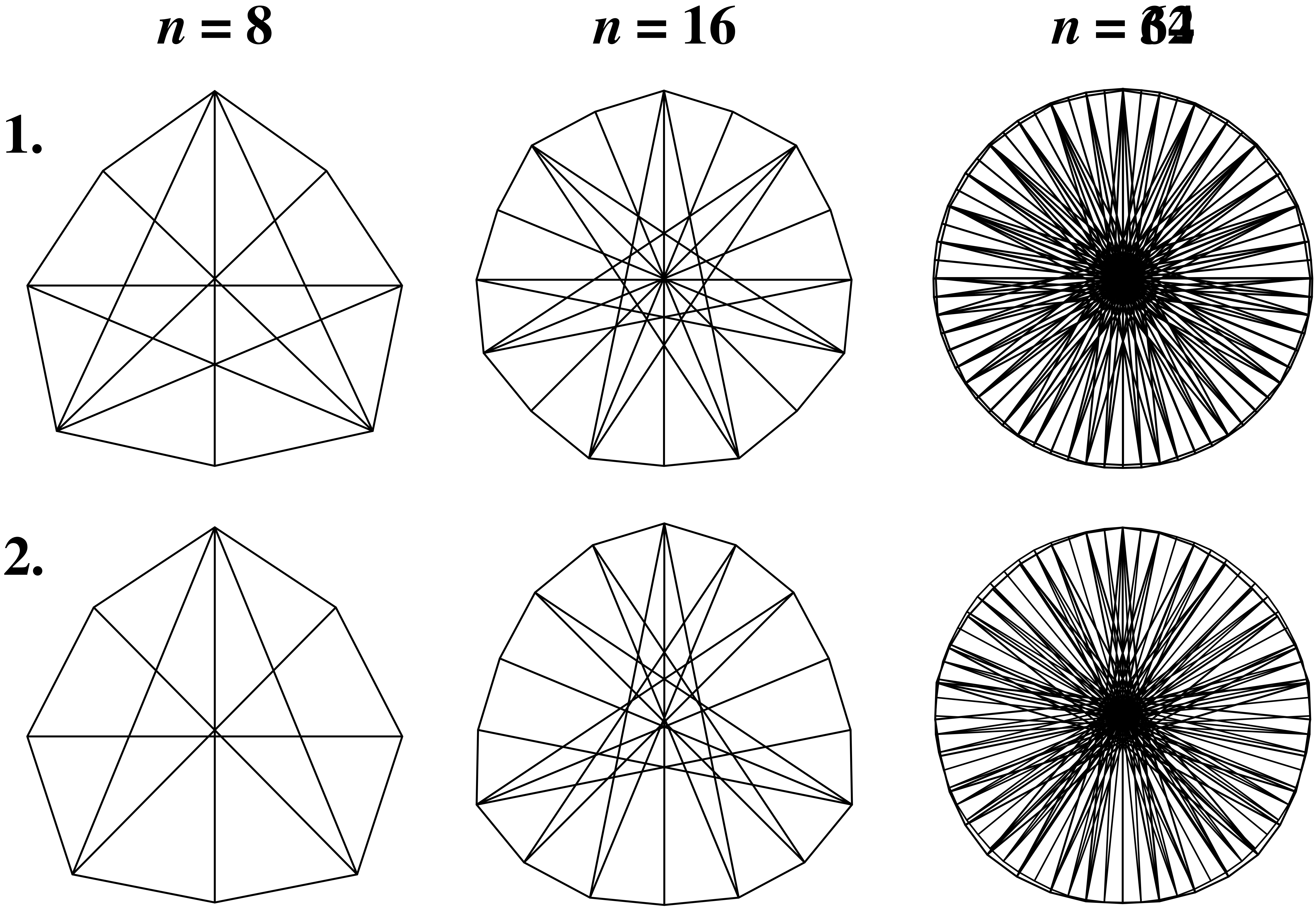
- Area of each polygon:
  - Exceeds regular $\boldsymbol{n}$-gon by $\sim \dfrac{\boldsymbol{\pi^3}}{\mathbf{16}\boldsymbol{n^2}}$.
  - Distance to upper bound: $< \dfrac{\mathbf{2}\boldsymbol{\pi^3}}{\mathbf{17}\boldsymbol{n^3}}$.

Reinhardt's Results: Perimeter

- $L \leq 2n \sin(\pi/2n)$.
- Attained precisely when n has an odd prime divisor.
- Regular n-gon optimal only for odd n.
- For $n \neq 2^m$:
  - Optimal n-gon unique only if $n = p,\ 2p$.
  - Optimal polygons are equilateral.

Open Questions: Perimeter

1. Optimal perimeter for 2^m-gon?
   - $m = 3$: Audet et al., 2007.
   - $m \geq 3$: M., 2006.
2. Best equilateral 2^m-gon?
   - $m = 3$: Vincze, 1950; Audet et al., 2004.
   - $m \geq 3$: M., 2008.
3. How many optimal n-gons?


n = 8
n = 16
n = 32
n = 64
1.
2.

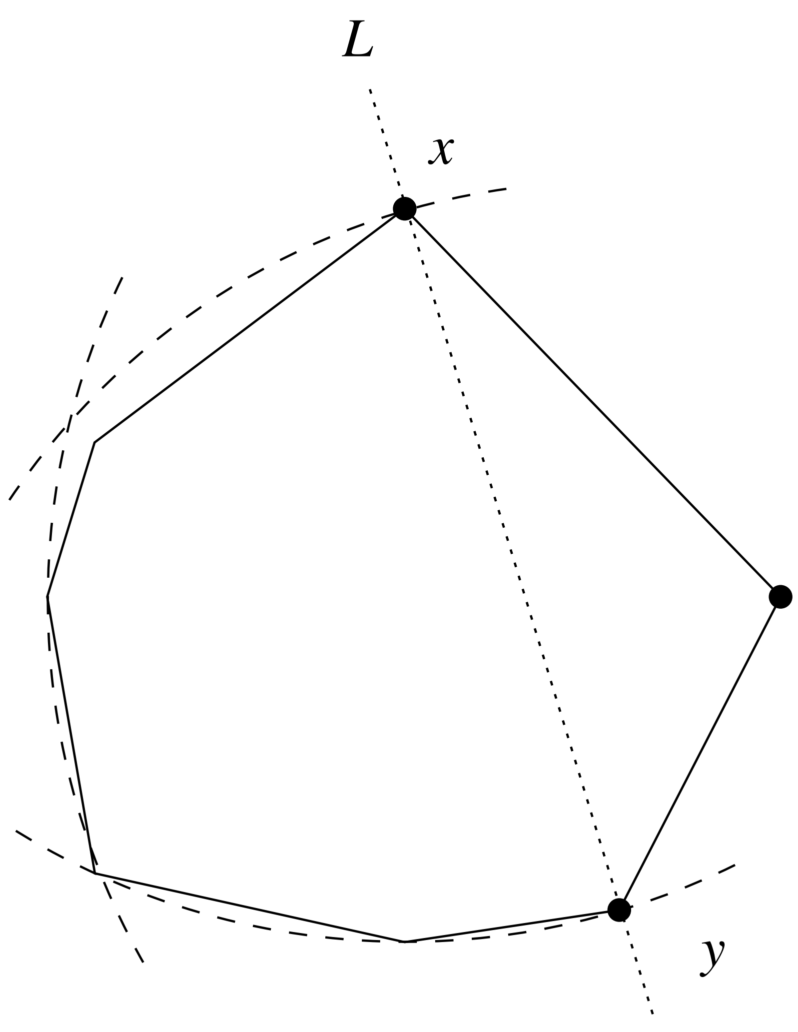
	Improvement	Proximity
1.	$\sim \dfrac{\pi^3}{8n^2}$	$\sim \dfrac{\pi^5}{16n^5}$
2.	$\sim \dfrac{\pi^3}{8n^2}$	$\sim \dfrac{3\pi^4}{n^4}$

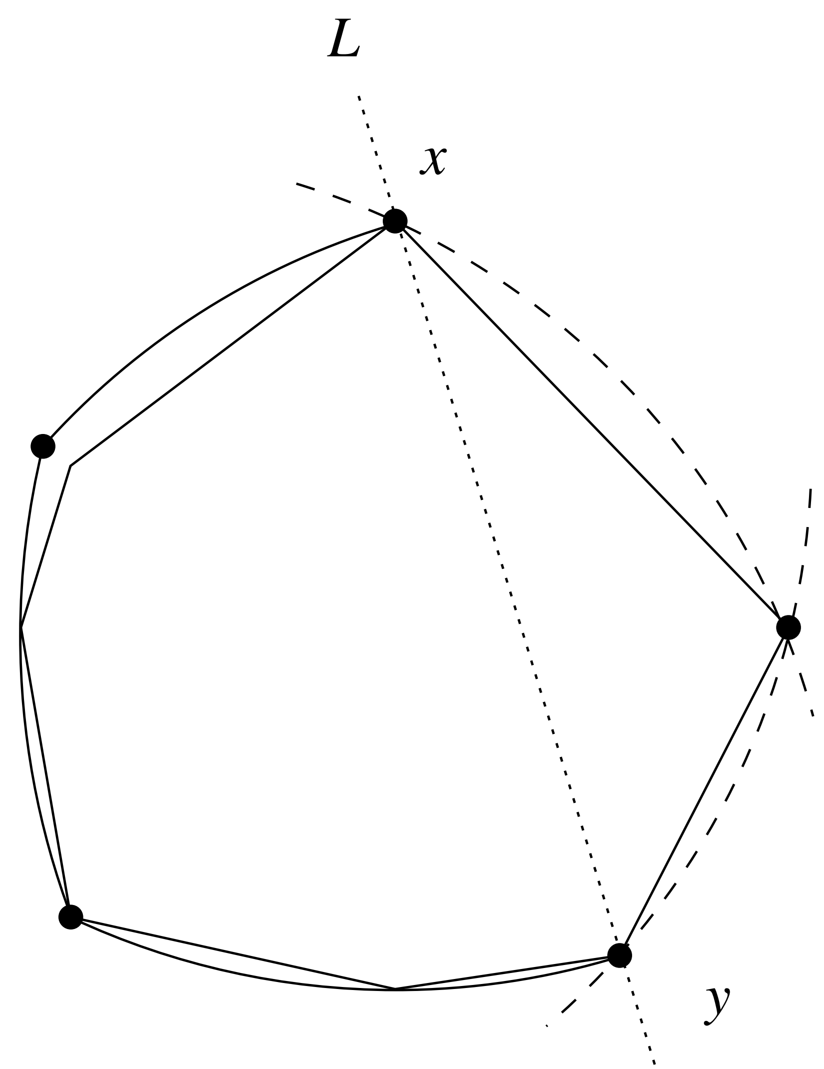
Problem 3: How Many Polygons?

- Reinhardt, 1922.
  - Some special cases.
  - "a purely combinatorial problem, which need not concern us further here."
- Datta, 1997.
  - Rediscovered Reinhardt results.
  - Few quantitative results: "at least as many maximizers as there are odd divisors (> 1) of n."

Constructing Optimal Polygons

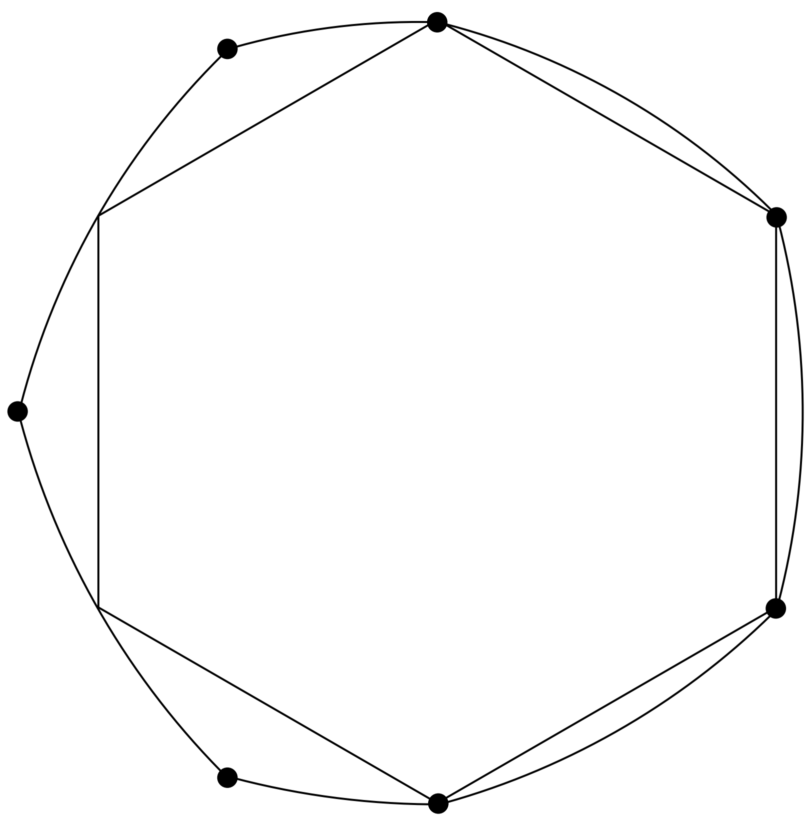
- Given a polygon ***P*** with unit diameter.
- Construct a *Reuleaux polygon* ***R*** with unit diameter containing it.





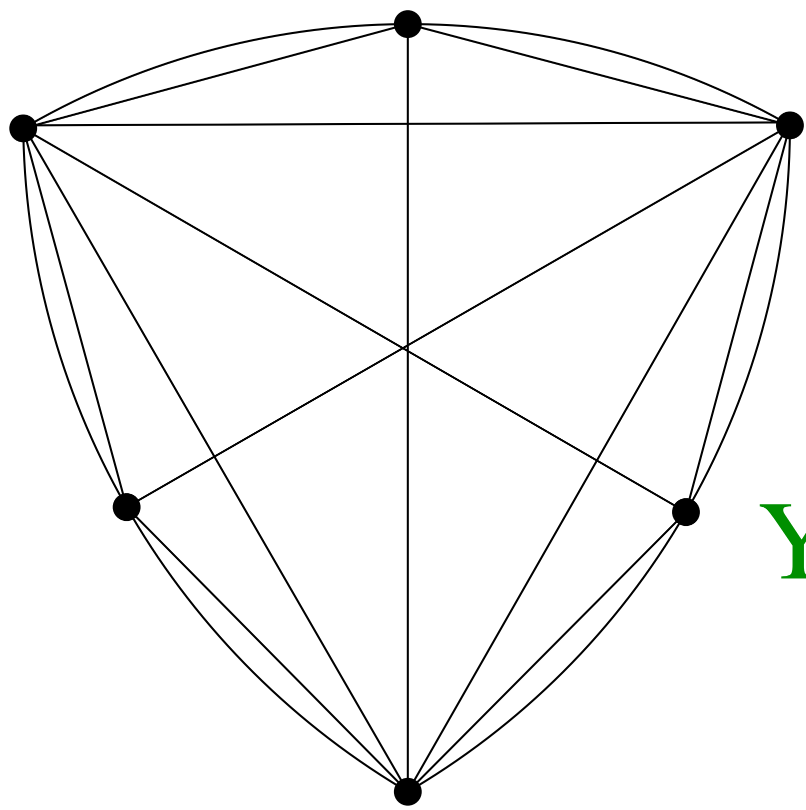
- Maximal perimeter when:
    - ***P*** is equilateral, and
    - each vertex of ***R*** is a vertex of ***P***.
- Example: $n = 6$.

Regular hexagon.



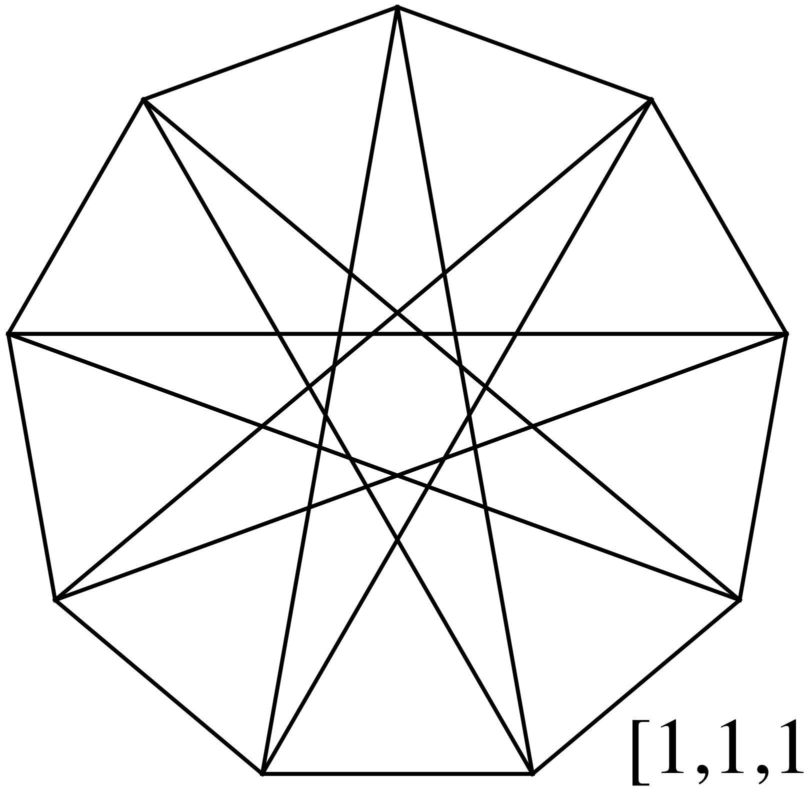
No!

Perturbed triangle.

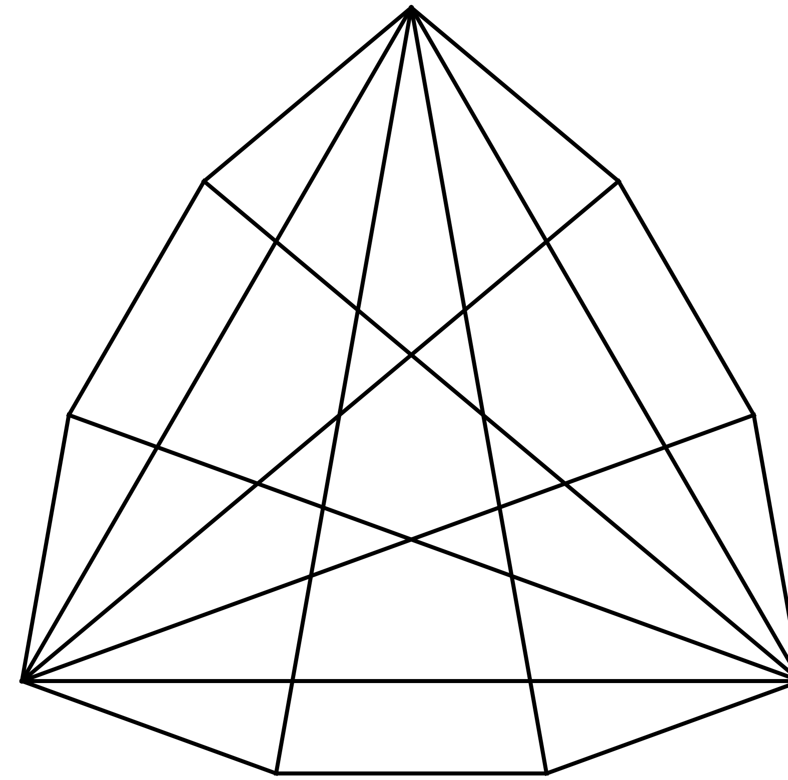


Yes!

n = 9: Two Polygons.

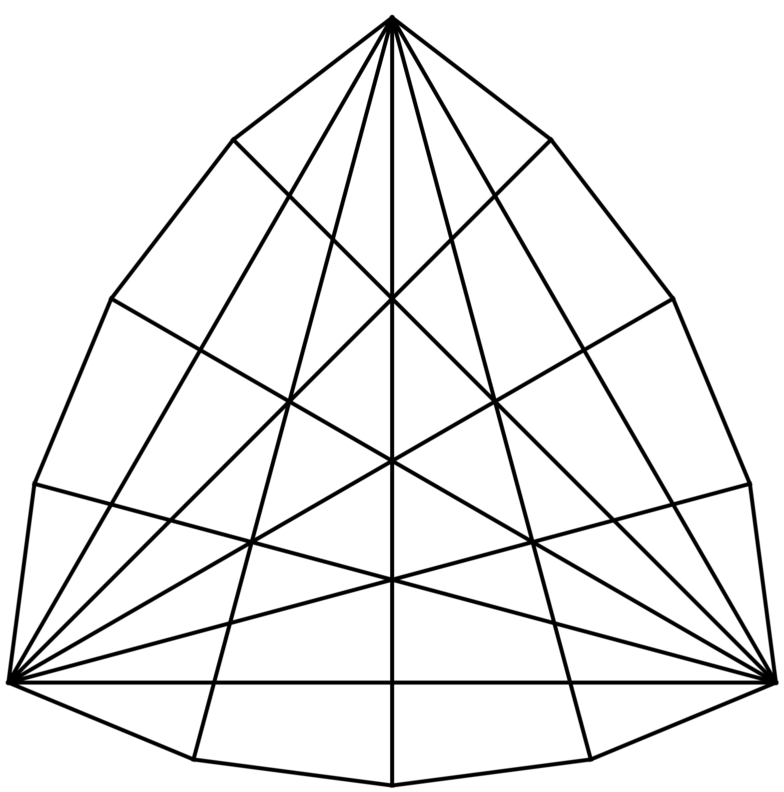


[1,1,1,1,1,1,1,1,1]

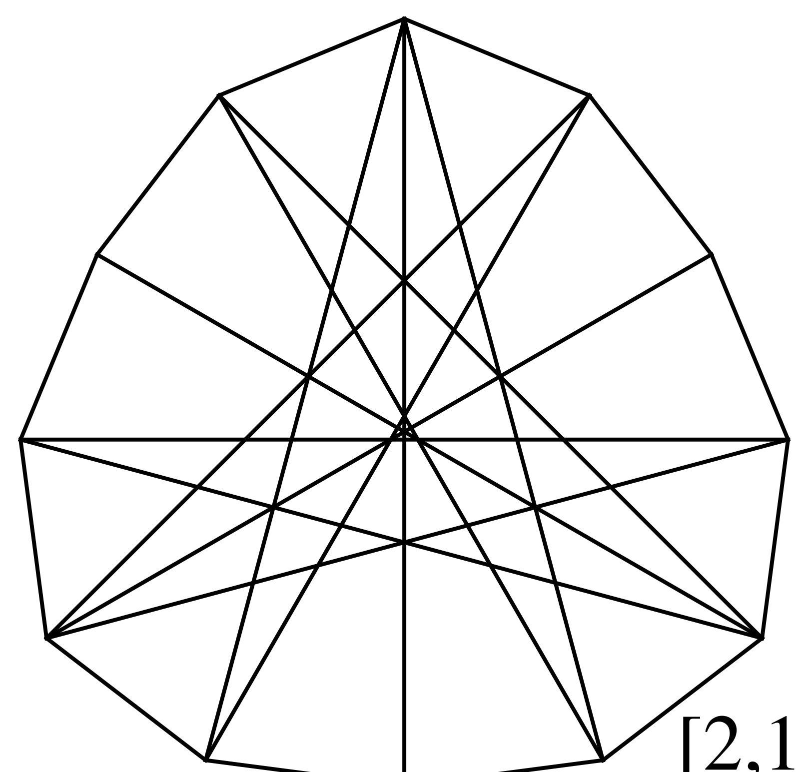


[3,3,3]

n = 12: Two Polygons.

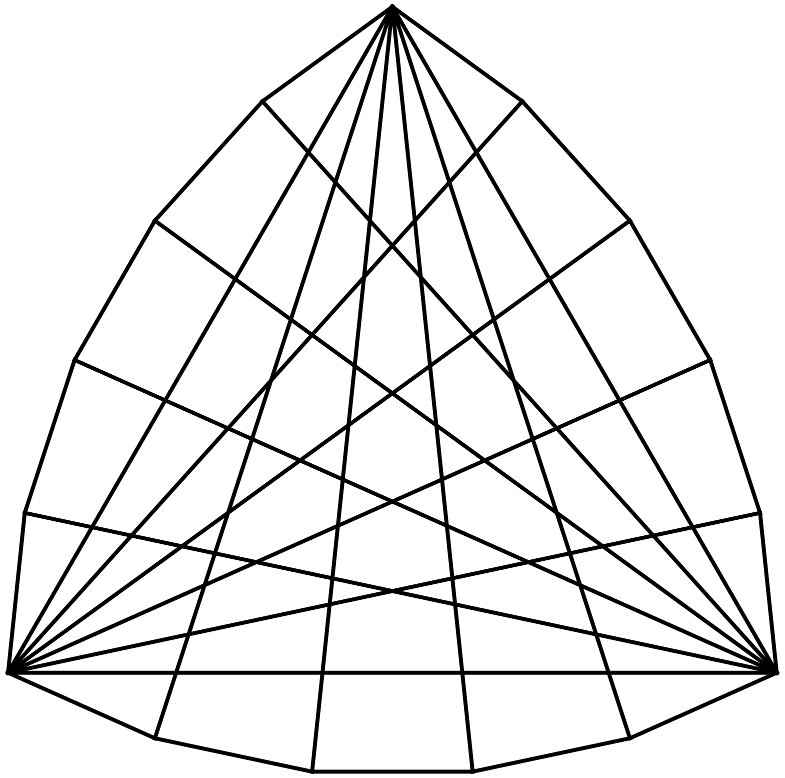


[4,4,4]

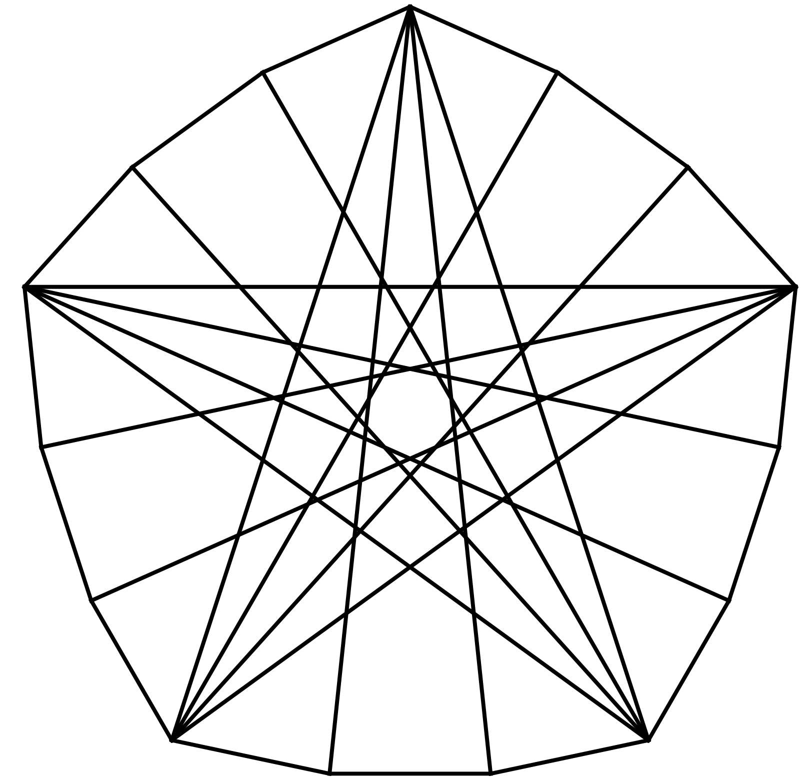


[2,1,1,2,1,1,2,1,1]

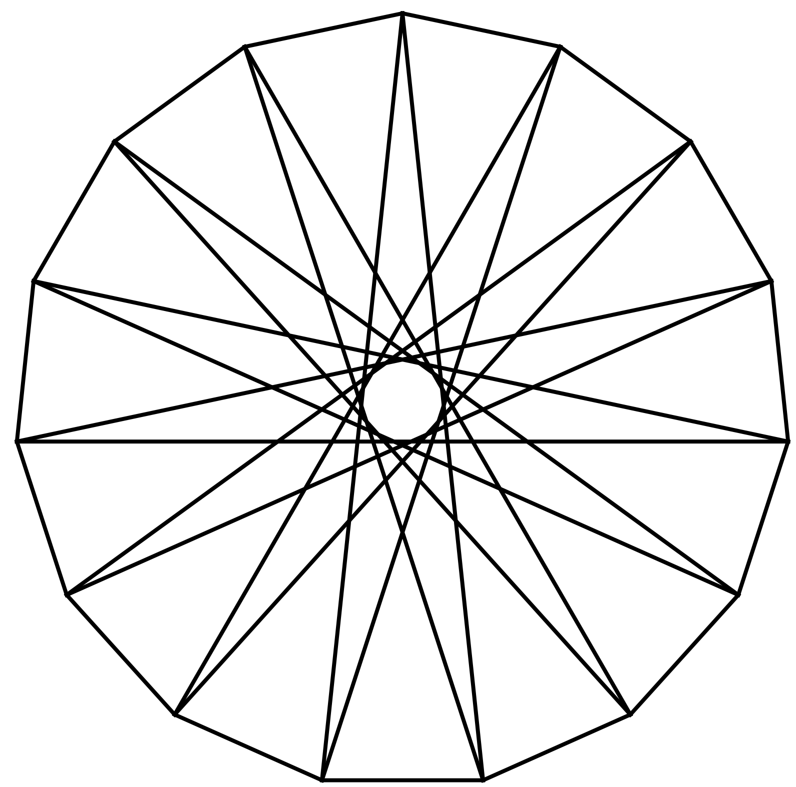
$n = 15$: Five Polygons.



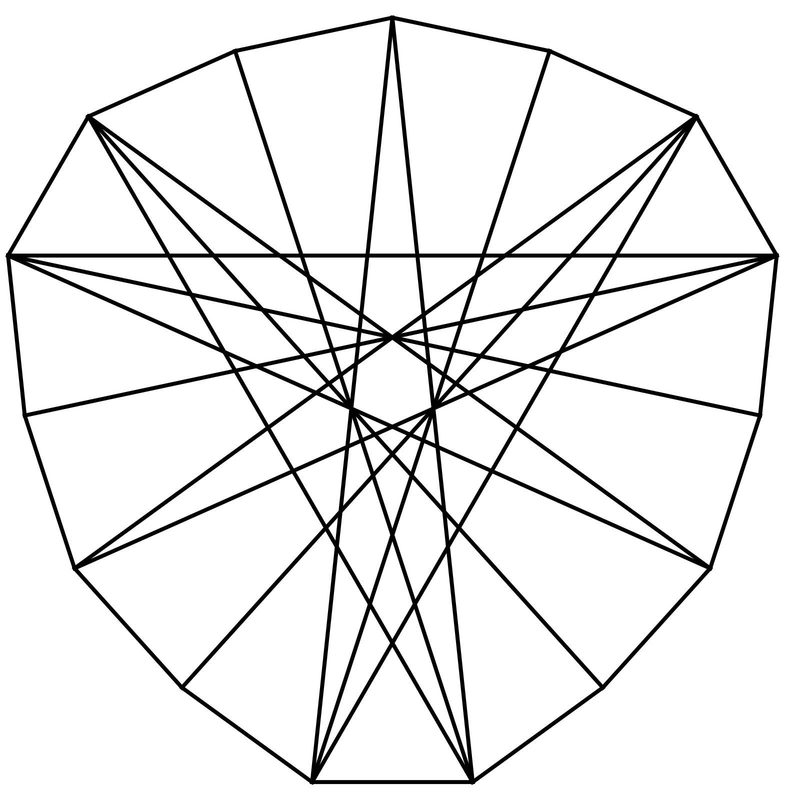
[5,5,5]



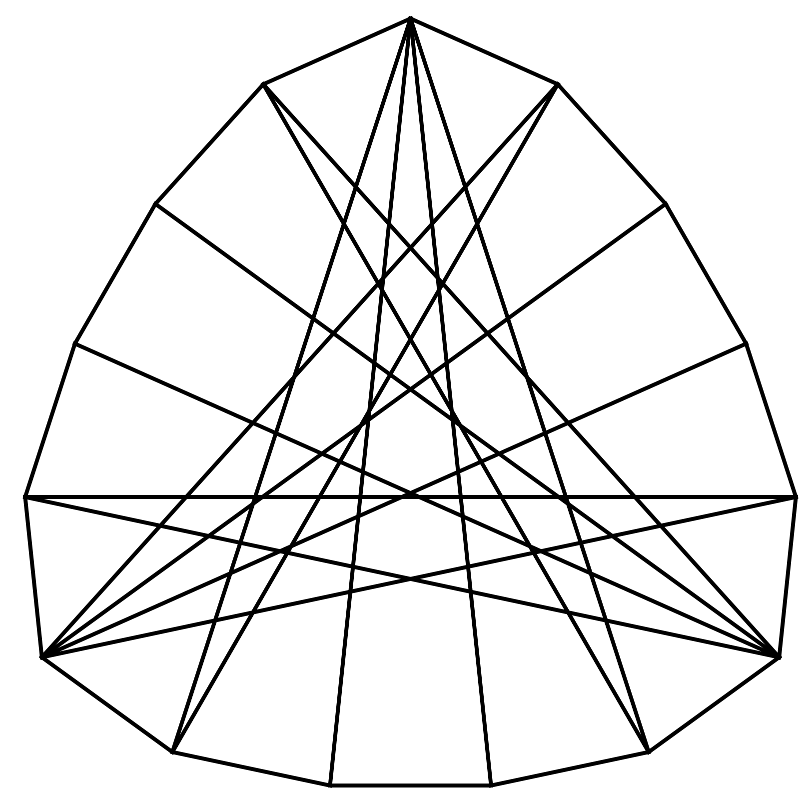
[3,3,3,3,3]



[1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1]



[2,2,1,2,2,1,2,2,1]

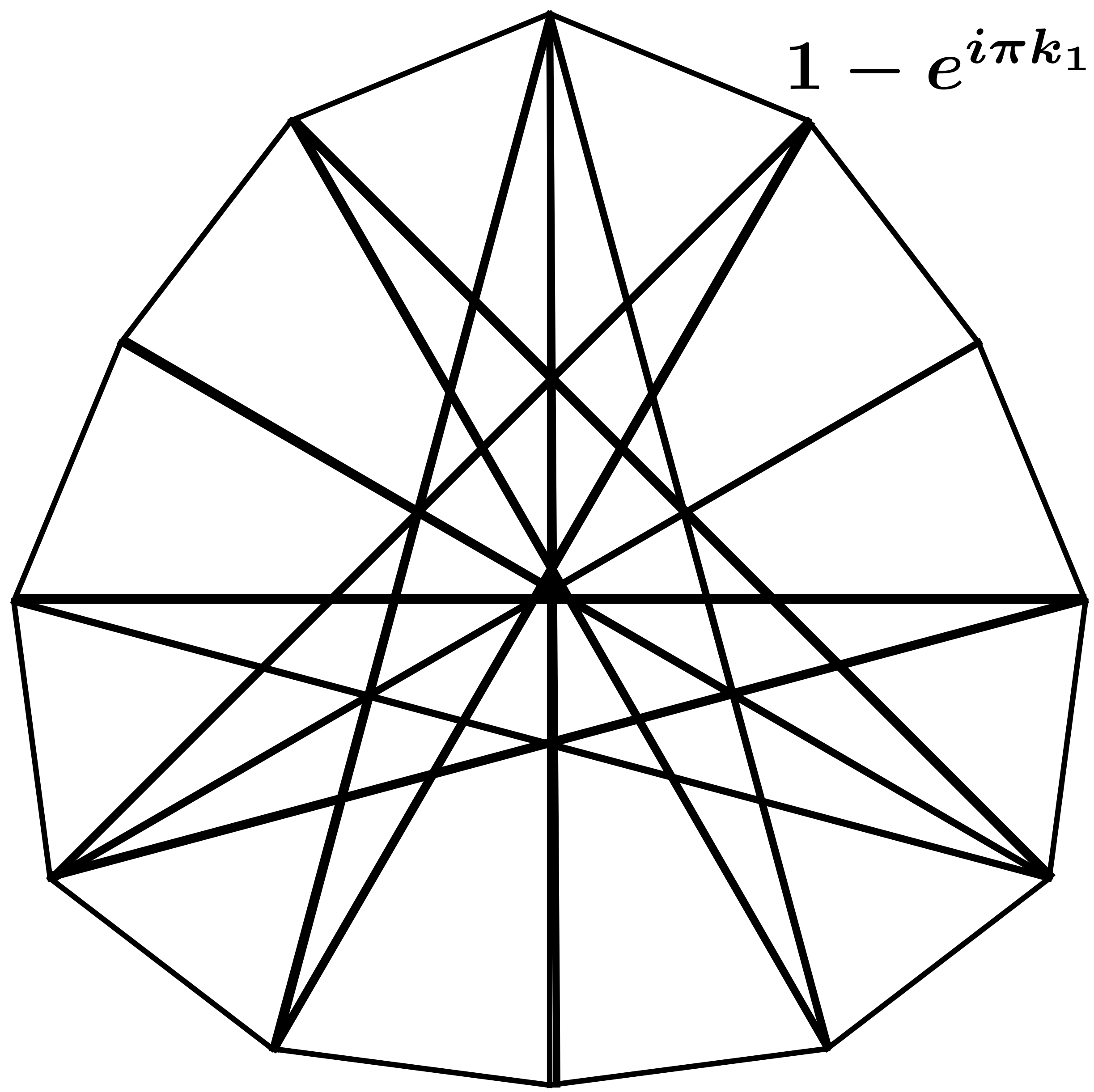


[3,1,1,3,1,1,3,1,1]

Construction

- Fix $n \geq 3$. Each optimal polygon $\boldsymbol{P}$ arises from:
  - Reuleaux polygon $\boldsymbol{R}$ with r bounding arcs.
  - r odd, $3 \leq r \leq n$.
  - Each arc has length $\pi k_i / n$, $\sum k_i = n$.
  - Create $\boldsymbol{P}$ from $\boldsymbol{R}$ by making k_i edges on arc i.
  - $\boldsymbol{P}$ specified by $[k_1, k_2, \ldots, k_r]$.
  - Equivalence classes: dihedral symmetry.

Example: Construct $\boldsymbol{P}$ for [1,2,1,1,2,1,1,2,1].



Requires:

$$\mathbf{1} - e^{i\pi k_1/n} + e^{i\pi(k_1+k_2)/n} - \cdots + e^{i\pi(k_1+\cdots+k_{r-1})/n} = \mathbf{0}.$$

Corresponding Polynomial Problem

- Optimal polygons correspond to polynomials $F(z)$ satisfying:
  - $\deg(F) < n$.
  - $F(0) = 1$.
  - Nonzero coefficients of $F(z)$ alternate ± 1.
  - Odd number of terms.
  - $\Phi_{2n}(z) \mid F(z)$.

For Example

$$x^{54} - x^{53} + x^{52} - x^{51} + x^{44} - x^{43} + x^{42} - x^{41} + x^{40} - x^{33} + x^{32} - x^{31} + x^{30} - x^{29} + x^{22} - x^{21} + x^{20} - x^{19} + x^{18} - x^{11} + x^{10} - x^{9} + x^{8} - x^{7} + 1$$

[7,1,1,1,1,7,1,1,1,1,7,1,1,1,1,7,1,1,1,1,7,1,1,1,1]

Periodic Case

- Suppose $[k_1, k_2, \ldots, k_r]$ is periodic, with odd period.
- $[k_{1,\ldots,}k_s,\ k_{1,\ldots,}k_s, \ldots, k_{1,\ldots,}k_s]$, s odd.
- Repeats d times, $d \geq 3$ odd.
- Construct $f(z)$ from $[k_1, \ldots, k_s]$.
- Then $F(z) = f(z) \cdot (z^n + 1)/(z^d + 1)$.
- So $\Phi_{2n}(z) \mid F(z)$.
- Let Per(n) denote the number of optimal polygons constructed in this way.

Constructing Periodic Polygons

- For each odd prime divisor p of n:
  - Construct all compositions $[k_{1,} \ldots, k_s]$ of $m := n/p$ with s odd.
  - Count equivalence classes under dihedral action.
  - Recall number of compositions of m is 2^{m-1}.
  - Number of compositions of m into an odd number of parts: 2^{m-2}.
  - Estimated number of equivalence classes: $\approx 2^{m-2}/m$.
  - Guess $\mathbf{Per}(n) \approx \frac{p}{4n} 2^{n/p}$, with p the smallest prime divisor of n.

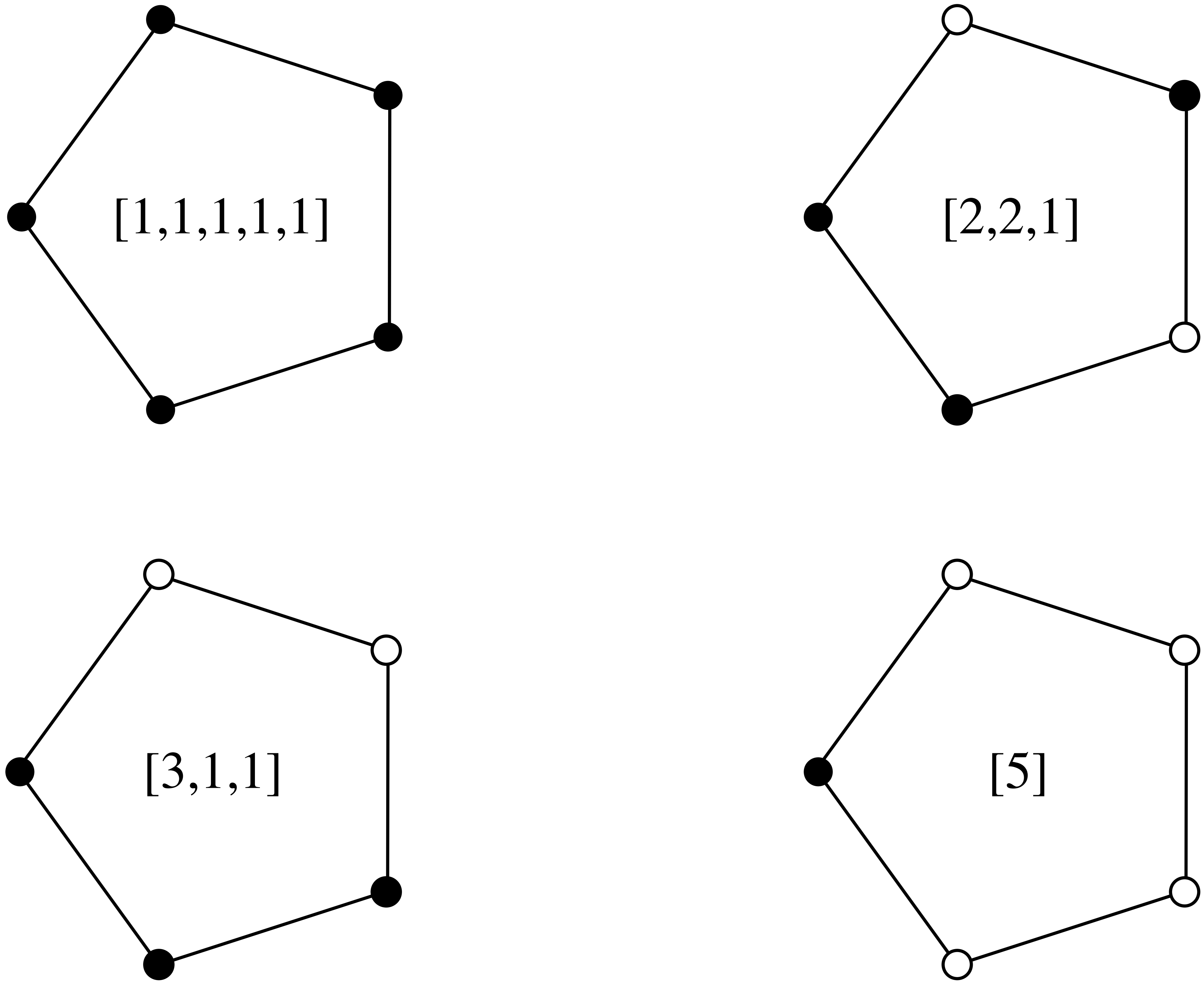
Accounting for Symmetry

- Equivalence classes of compositions of m into an odd number of parts.

$$\updownarrow$$

- Equivalence classes of m-bead necklaces with:
  - black and white beads, and
  - an odd number of black beads.
- Let $G(m)$ denote the number of such equivalence classes.

Example: $G(5) = 4$.



Results: $n = pq$

- If $m = q$ with q an odd prime, then

$$G(q) = \frac{2^{q-1} + q - 1}{2q} + 2^{(q-3)/2}.$$

- If $n = pq$ then

$$\mathrm{Per}(n) = G(p) + G(q) - 1.$$

- $\mathrm{Per}(15) = G(3) + G(5) - 1 = 2 + 4 - 1 = 5.$

- $\mathrm{Per}(55) = 63 + 4 - 1 = 66.$

Results: $n = p^r$

- If $n = p^r$, so $m = p^{r-1}$, then

$$\mathrm{Per}(n) = \frac{1}{m}\left(2^{m-1} + (p-1)\sum_{i=1}^{r} p^{i-1} 2^{mp^{-i}-1}\right) + 2^{(m-3)/2}.$$

- $\mathrm{Per}(3) = 1,\ \mathrm{Per}(9) = 2,\ \mathrm{Per}(27) = 23,$

$\mathrm{Per}(81) = 1\,246\,863,$

$\mathrm{Per}(243) = 74\,62505\,05989\,93194\,49231,$

$\mathrm{Per}(729) = 1\,45419\,51150\,43937\,71982\,16440\,33533\,26640\,10935\,68217\,89588\,64023\,40717\,15229\,83567.$

Theorem: Assume $\boldsymbol{n}$ is not a power of $\mathbf{2}$, and let $\boldsymbol{p} > \mathbf{1}$ be its smallest odd divisor. Then

$$\mathbf{Per}(\boldsymbol{n}) \geq \frac{\boldsymbol{p}}{\mathbf{4}\boldsymbol{n}} \mathbf{2}^{\boldsymbol{n}/\boldsymbol{p}}.$$

In fact, for fixed $\boldsymbol{p}$ and large $\boldsymbol{n}$,

$$\mathbf{Per}(\boldsymbol{n}) \sim \frac{\boldsymbol{p}}{\mathbf{4}\boldsymbol{n}} \mathbf{2}^{\boldsymbol{n}/\boldsymbol{p}}.$$

Example:

$$\mathbf{Per}(105) = \mathbf{245\,518\,324},$$

$$\mathbf{Per}(105) \geq \mathbf{245\,426\,703}.$$

Aperiodic Case

- Can we construct suitable $F(z)$ with aperiodic $[k_1, \ldots, k_r]$?

- Let $\mathbf{Aper}(n)$ denote the number of such "sporadic" solutions.

- Fact: The ideal $(\Phi_n(z))$ in $\mathbb{Z}[x]$ is generated by the polynomials

$$\left\{ \Phi_p(z^{n/p}) = \frac{z^n - 1}{z^{n/p} - 1} : p \mid n \right\}.$$

- Suppose $p_1, \ldots, p_k$ are the prime divisors of n.

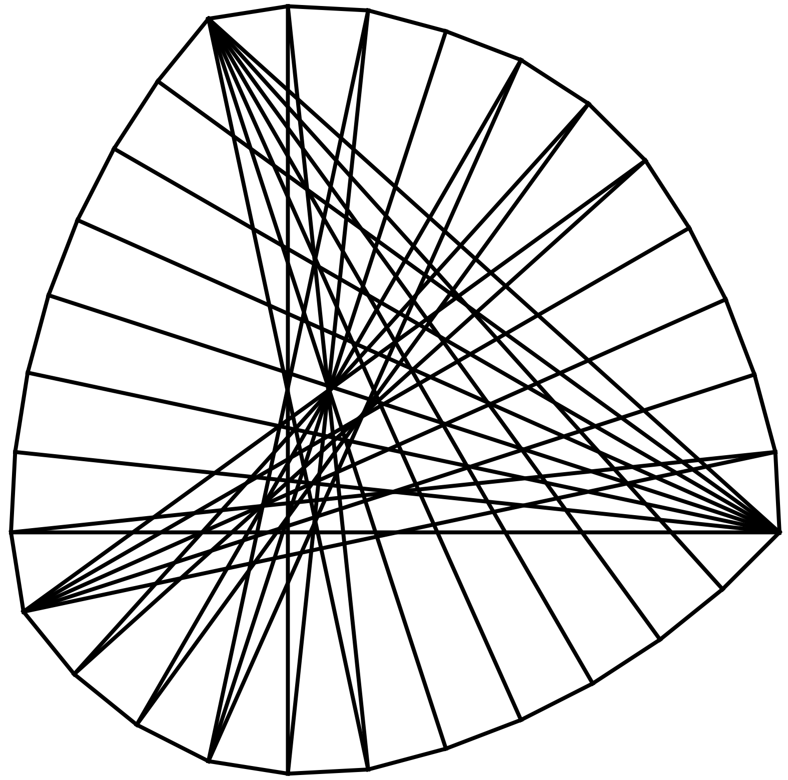
- In general, we need $F(z)$ with $\deg(F) < n$, nonzero coefficients alternate ± 1, and

$$F(z) = f_{p_1}(z)\Phi_{p_1}(-z^{n/p_1}) + \cdots + f_{p_k}(z)\Phi_{p_k}(-z^{n/p_k}).$$

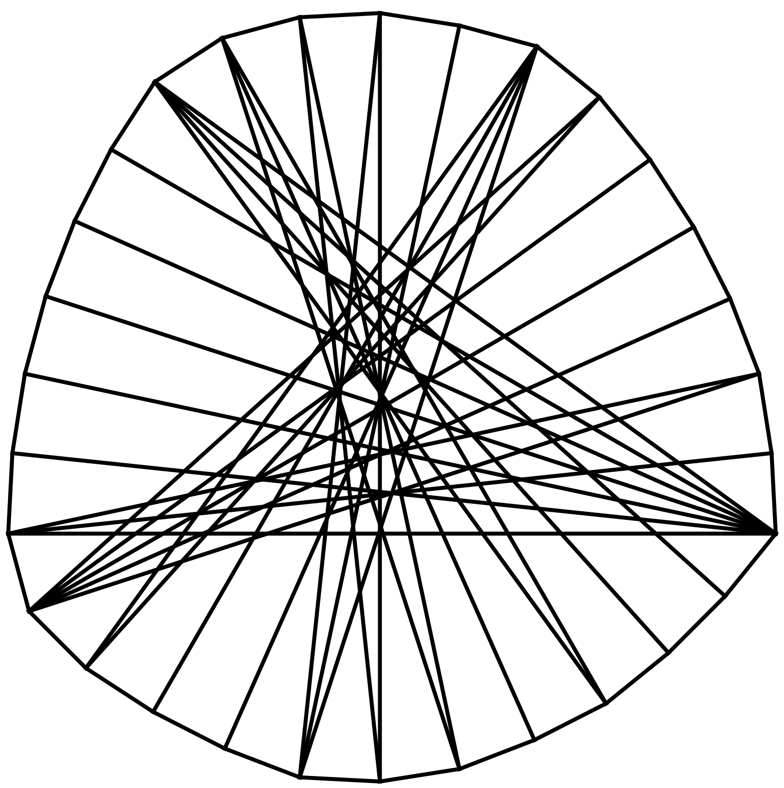
- Periodic case: each $f_{p_i}(z) = 0$, except one.

- Can any other combinations produce valid $F(z)$?

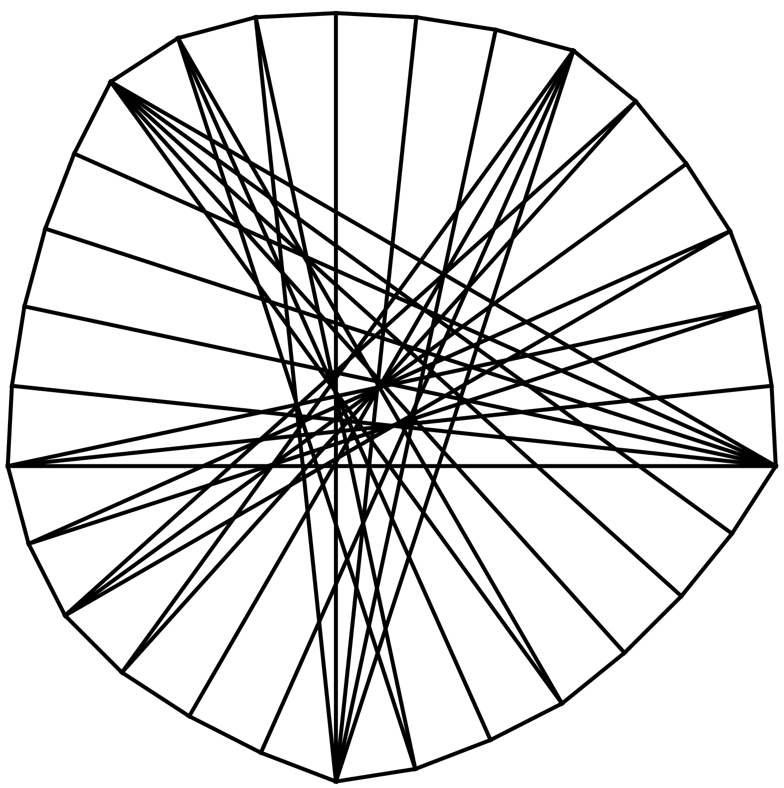
Yes! n = 30:



[7, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 4, 1, 1]

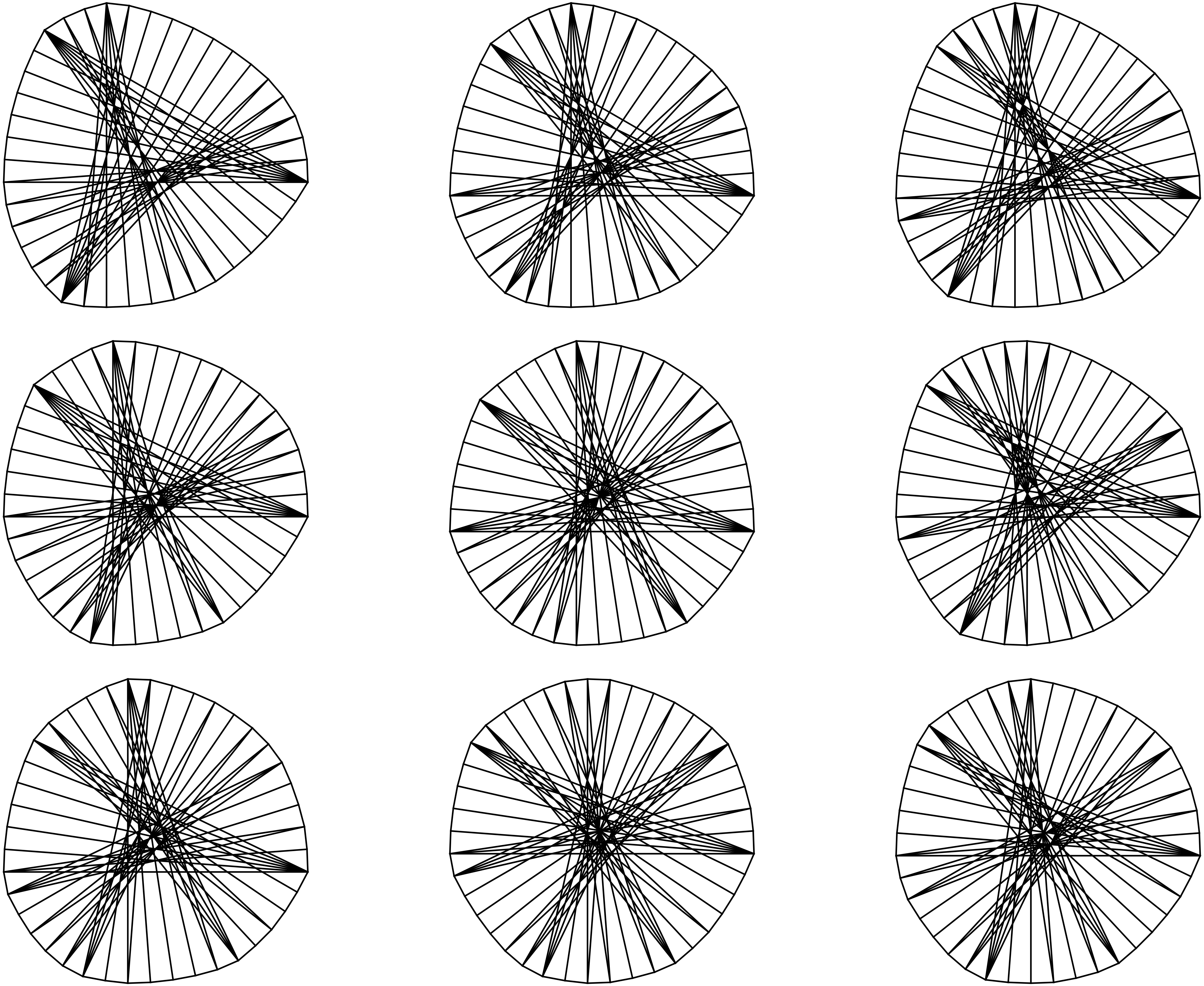


[6, 3, 1, 2, 1, 1, 1, 1, 2, 3, 1, 1, 4, 1, 2]



[5, 4, 1, 2, 1, 1, 4, 3, 1, 1, 2, 1, 1, 1, 2]

42



Results

Theorem: There are no aperiodic polygons if $\boldsymbol{n = 2^a p^b}$.

Conjecture: None if $\boldsymbol{n = pq}$.

That is, if $\boldsymbol{\Phi_{2pq}(x) \mid F(x)}$, $\boldsymbol{\deg(F) < pq}$, and the nonzero coefficients of $\boldsymbol{F}$ alternate $\boldsymbol{\pm 1}$, then $\boldsymbol{\Phi_p(-x^q) \mid F(x)}$ or $\boldsymbol{\Phi_q(-x^p) \mid F(x)}$.

Empirically: Aperiodic polygons occur everywhere else.

Data

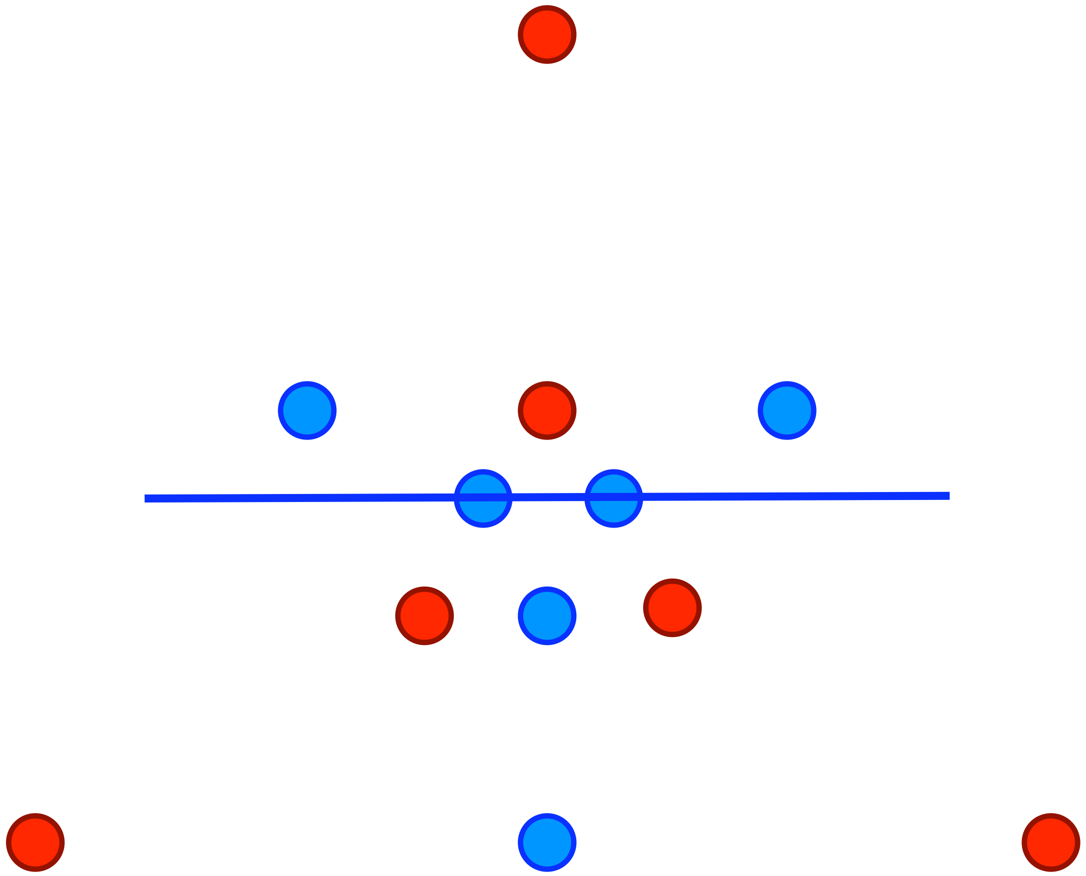
n	Per(*n*)	Aper(*n*)
30	38	3
42	329	9
45	633	144
60	13,464	4392
63	25,503	1308
66	48,179	93
70	358	27
75	338,202	153,660
78	647,330	315
84	2,400,942	161,028

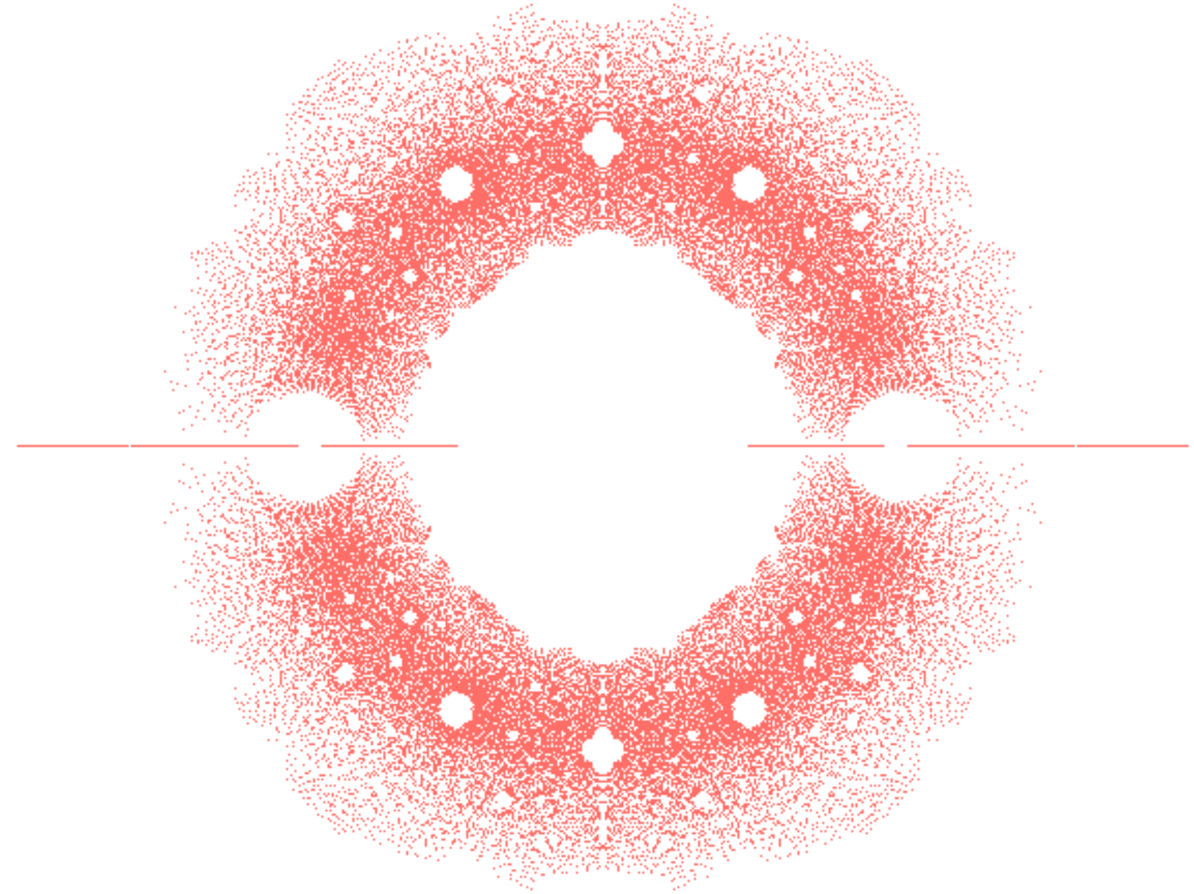
More Data

n	Per(n)	Aper(n)
90	8,959,826	5,385,768
99	65,108,083	192,324
102	126,355,340	3855
110	48,487	279
114	1,808,538,359	13,797
130	648,304	945
140	3,047,732	633,528
154	49,336	837
170	126,355,369	11,565
182	650,485	2835

In Conclusion . . .

Some New Perspectives . . .





Thanks!