An extreme polynomial with two variables

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Reconstruction of words

Definitions

- alphabet: $\Sigma = \{0, 1, ..., t-1\}$
- word: $w = w_1 ... w_n \in \Sigma^n$
- u is a subword of w if $u_i = w_{r(i)}$ for each i and $1 \leq r(1) < r(2) < \cdots < r(k) \leq n$
- $s_k(w)$: the multiset of the k-letter subwords of w; $|s_k(w)| = \binom{n}{k}$
- w is reconstructible from $s_k(w)$ means that w is uniquely determined by $s_k(w)$

Reconstruction of words

Problem (Kalashnik)

Given n, determine the minimal k such that every word w of length at most n is reconstructible from $s_k(w)$.

Theorem (Kalashnik, Manvel)

Every word of length n is reconstructible from $s_k(w)$ if $k \geq \lfloor \frac{n}{2} \rfloor$.

Theorem (Krasikov és Roditty)

Every word of length n is reconstructible from $s_k(w)$ if $k \geq \lfloor \frac{16}{7}\sqrt{n} \rfloor + 5$.

Note: it's enough to work over the alphabet $\{0, 1\}$

Reconstruction of words

The method of Krasikov and Roditty: consider the vector

$$S(w) = \sum_{v \in s_k(w)} v$$ (subword sum); if $w = w_0 \dots w_{n-1}$ then

the jth position of $S(w)$ is $\sum_{i=0}^{n-1} \binom{i}{j}\binom{n-i-1}{k-j-1} w_i \qquad j = 0, \dots, k-1$

If $u = u_0 \dots u_{n-1}$, $s_k(w) = s_k(u)$ then $S(w) = S(u)$ so

$\sum_{i=0}^{n-1} \binom{i}{j}\binom{n-i-1}{k-j-1} w_i = \sum_{i=0}^{n-1} \binom{i}{j}\binom{n-i-1}{k-j-1} u_i$ for $j = 0, \dots, k-1$.

Define $f_j(x) = \binom{x}{j}\binom{n-x-1}{k-j-1}$, deg$= k-1$, $j = 0, \dots, k-1$ then $\{f_j : j = 0, \dots, k-1\}$ is a basis of polynomials of deg$\leq k-1$

So for any $\phi(x)$ of deg$\leq k-1$: $\sum_{i=0}^{n-1} w_i \phi(i) = \sum_{i=0}^{n-1} u_i \phi(i)$

Choosing $\phi(x) = x, x^2, \dots, x^{k-1}$ and deleting terms with zero coefficient we get that for some s the system of equations

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Choosing $\phi(x) = x, x^2, \dots, x^{k-1}$ and deleting terms with zero coefficient we get that for some s the system of equations

$$\begin{gathered}
a_1 + a_2 + .. + a_s = b_1 + b_2 + .. + b_s \\
a_1^2 + a_2^2 + .. + a_s^2 = b_1^2 + b_2^2 + .. + b_s^2 \\
\vdots \\
a_1^{k-1} + a_2^{k-1} + .. + a_s^{k-1} = b_1^{k-1} + b_2^{k-1} + .. + b_s^{k-1} \\
a_1 < a_2 < .. < a_s, \quad b_1 < b_2 < .. < b_s
\end{gathered}$$

has a non-trivial solution with $a_i, b_i \in [0, n-1]$:
a_i: positions of “1” in w, b_i: positions of “1” in u
(Trivial solution: $a_i = b_i$ for all $i \quad \longleftrightarrow \quad w = u$)

(Prouhet-Tarry-Escott problem)

Reconstruction of words

Theorem (P. Borwein, T. Erdélyi, G. Kós)

For $k \geq \lfloor \frac{16}{7}\sqrt{n} \rfloor + 5$, the P-T-E problem has only trivial solutions.

$\Downarrow$

Theorem (Krasikov and Roditty)

Every word of length n is reconstructible from $s_k(w)$ if $k \geq \lfloor \frac{16}{7}\sqrt{n} \rfloor + 5$.

This method cannot give more: the P-T-E problem has a non-trivial solution if $k \leq \sqrt{2\log 2}\sqrt{\frac{n}{\log n}} - \frac{1}{2}$

The best known lower bound is $e^{c\sqrt{\log n}}$ (Dudik, Schulman).

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Reconstruction of matrices

- square matrices: $A \in \Sigma^{n \times n}$ w.l.o.g. $\Sigma = \{0, 1\}$
- submatrices of size k: deleting $n - k$ rows and columns
- symmetric or arbitrary deletion $\longleftrightarrow \mathrm{sym}M_k(A)$ or $M_k(A)$
- no essential difference
- $\mathrm{sym}A_k^+ = \sum_{B \in \mathrm{sym}M_k(A)} B$ or $A_k^+ = \sum_{B \in M_k(A)} B$ (submatrix sums)

Problem

Given n, determine the smallest $k = (\mathrm{sym})k_{\mathrm{min}}$, such that every $A \in \Sigma^{n \times n}$ matrix is reconstructible from $(\mathrm{sym})M_k(A)$.

Problem

Given n, determine the smallest $k = (\mathrm{sym})k^+_{\mathrm{min}}$, such that every $A \in \Sigma^{n \times n}$ matrix is reconstructible from $(\mathrm{sym})A_k^+$.

Reconstruction of matrices

Note that the lower bound on words implies
$e^{c\sqrt{\log n}} < k_{\min} \le \mathrm{sym}k_{\min}$.

Theorem (G. Kós, P. Ligeti, P.Sz.)

In $\Sigma^{n\times n}$ (n large) we have

$$c \cdot \frac{n^{2/3}}{\sqrt[3]{\log n}} \le k^+_{\min} \le \mathrm{sym}k^+_{\min} \le 38n^{2/3}.$$

Theorem (G. Kós, P. Ligeti, P.Sz.)

If $k > 38n^{2/3}$ and n is large enough then every $A \in \Sigma^{n\times n}$ is uniquely determined by $M_k(A)$ as well as by $\mathrm{sym}M_k(A)$, i.e. $k_{\min} \le \mathrm{sym}k_{\min} \le 38n^{2/3}$.

Reconstruction of matrices

Theorem

In $\Sigma^{n\times n}$ we have

$$c\cdot\frac{n^{2/3}}{\sqrt[3]{\log n}}\leq k^{+}_{\min}\leq \mathrm{sym}k^{+}_{\min}\leq 38n^{2/3}.$$

- $k^{+}_{\min}\leq \mathrm{sym}k^{+}_{\min}$
- $k_{\min}\leq k^{+}_{\min}$
- $k_{\min}\lneq k^{+}_{\min}$??? (we don't know it for words either)
- we consider here the "non-symmetric" case

The lower bound

Pigeon hole principle:

Theorem

For any $A \in \Sigma^{n\times n}$,

$$c \cdot \frac{n^{2/3}}{\sqrt[3]{\log n}} \le k^+_{\min} \le \mathrm{sym}k^+_{\min} \le 38n^{2/3}$$

holds.

(There are 2^{n^2} $n \times n$ matrices and at most $\left(\binom{n}{k}^2 + 1\right)^{k^2}$ possible A_k^+ matrices.)

Translation to the language of polynomials

Lemma

Let $S_{uv}(A)$ denote the number of 1-s in the submatrices in row u and column v, so the (u,v)-th entry in the sum-matrix A_k^+. Then

$$S_{uv}(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_u(i)\beta_v(j)a_{ij},$$

where $\beta_u(x) = \binom{x-1}{u-1}\binom{n-x}{k-u}$.

Lemma

(i) The polynomials $\{\beta_u(x), 1 \leq u \leq k\}$ form a base for polynomials of $\deg < k$.
(ii) The polynomials $\{\beta_u(x)\beta_v(y), 1 \leq u, v \leq k\}$ form a base for bivariate polynomials of $\deg_x, \deg_y < k$.

Translation to the language of polynomials

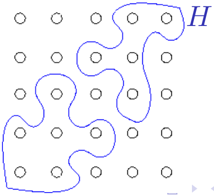
Theorem

Let $A \neq B \in \Sigma^{n \times n}$, *for which* $M_k(A) = M_k(B)$, *let* $\delta_{ij} = a_{ij} - b_{ij}$. *Then for every* $r(x, y)$ *with* $\deg_x, \deg_y \leq k - 1$

$$\sum_{x=1}^{n} \sum_{y=1}^{n} \delta_{xy} \cdot r(x, y) = 0.$$

- nonzero entries in a matrix $\longrightarrow$ subset H of an $n \times n$ lattice

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$



Constructing polynomials: upper bound

Lemma

For n large enough and for arbitrary $H \subset \{1, 2, \ldots \ldots, n\}^2$ there exists a point $\mathbf{a} = (a_1, a_2) \in H$ and a polynomial $p(x, y)$ for which $\deg_x p, \deg_y p < 38n^{2/3}$ and

$$p(a_1, a_2) > \sum_{(x,y) \in H, \quad (x,y) \neq (a_1, a_2)} |p(x, y)|.$$

This immediately implies that (if $k > 38n^{2/3}$ then)

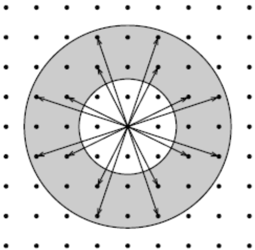
$$\sum_{x=0}^{n-1} \sum_{y=x}^{n-1} \delta_{xy} r(x, y) = 0$$

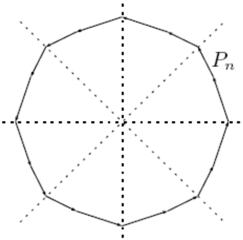
cannot hold if $r(x, y) = p(x, y)$, a contradiction, we are done.

Constructing polynomials: a lattice polygon

Lemma

For n large enough there exists a convex lattice polygon P_n for which
(a) $\{1, 2, \ldots, n\}^2 \subseteq P_n$;
(b) the side lengths of P_n lie in $[n^{1/3}, 2n^{1/3}]$;
(c) the sides of P_n do not contain further lattice points.



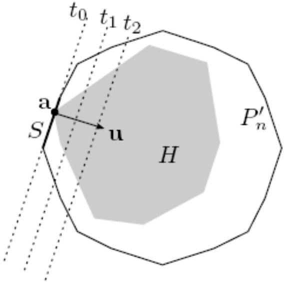


Constructing polynomials: Chebyshev polynomials

Lemma

For any $A, M \in \mathbb{R}_+$ *there exists an* $f_1(x) \in \mathbb{R}[x]$ *for which*
(a) $f_1(0) = M$*;*
(b) $|f_1(x)| \leq \min\left(M, \frac{1}{x^2}\right)$ *for all* $x \in (0, A]$*;*
(c) $\deg f_1 < \sqrt{\pi}\sqrt{A}\sqrt[4]{M} + 2$.

- $M = 19, A = 2n^{4/3}$
- $g_1(x, y) = u_1(x - a_1) + u_2(y - a_2)$
- $p_1(x, y) = f_1\big(g_1(x, y)\big)$

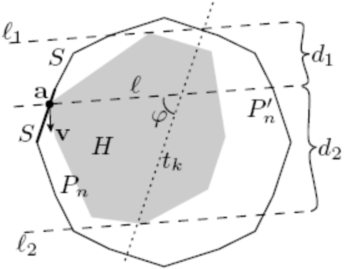


Constructing polynomials: Chebyshev polynomials

Lemma

For any $A, B, M \geq 1$ *there exists an* $f_2(x) \in \mathbb{R}[x]$ *for which*
(a) $f_2(0) = M$;
(b) $|f_2(x)| < \min\left(4M, \frac{1}{x^2}\right)$ *for all* $x \in [-A, B]$, $x \neq 0$;
(c) $\deg f_2 < 7\sqrt{ABM} + 2$.

- $A = \max(d_1, 1)$,
  $B = \max(d_2, 1)$
- $M = 1$, $\varphi = \text{arc } \sin n^{-1/3}$
- $g_2(x, y) = v_1(x - a_1) + v_2(y - a_2)$
- $p_2(x, y) = f_2\big(g_2(x, y)\big)$



Constructing the polynomial

$$p(x,y) := p_1(x,y) \cdot p_2(x,y).$$

Then $\deg p < 38n^{2/3}$ and

$$p(a_1,a_2) > \sum_{(x,y)\in H,\ \ (x,y)\neq(a_1,a_2)} |p(x,y)|.$$

Theorem

For any $A \in \Sigma^{n\times n}$

$$c \cdot \frac{n^{2/3}}{\sqrt[3]{\log n}} \le k^+_{\min} \le \mathrm{sym}k^+_{\min} \le 38n^{2/3}.$$

Thanks for your attention!