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MÜNTZ, REMEZ, NEWMAN

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Typeset by $\mathcal{AMS}$-TEX

1. Müntz Spaces, Müntz Polynomials

Notation

$$(\lambda_j)_{j=0}^{\infty}, \qquad 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$
$$= \left\{ p : p(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}, \quad a_k \in \mathbb{R} \right\}$$

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$$\|f\|_A := \sup_{x \in A} |f(x)|$$

Theorem (Müntz).

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0,1]$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Problem 1.1. *Characterize the compact sets*

$$A \subset [0, \infty)$$

for which "Müntz's Theorem holds", that is, for which

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C(A)$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Theorem 1.2. *Müntz's Theorem holds on every compact set $A \subset [0, \infty)$ of positive Lebesgue measure. That is, given a compact set $A \subset [0, \infty)$ of positive Lebesgue measure,*

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C(A)$ if and only if

$$(*) \qquad \sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Moreover, if (*) does not hold then every function $f \in C(A)$ from the uniform closure of

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

on A can be extended analytically throughout the disk

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < \text{ess sup } A\},$$

where

$$\text{ess sup } A := \sup\{x \ge 0 : m(A \cap [0, x]) > 0\}.$$

Problem 1.3. *Does Müntz's Theorem hold on every compact set* $A \subset [0, \infty)$ *of positive logarithmic capacity?*

Problem 1.4. *Does Müntz's Theorem hold on the ternary Cantor set?*

Problem 1.5. *Is there a compact set* $A \subset [0, \infty)$ *of Lebesgue measure* 0 *on which Müntz's Theorem holds?*

Let

$$\Lambda := (\lambda_j)_{j=0}^{\infty}$$

be a sequence of distinct real numbers. Let

$$R(\Lambda) := \left\{ \frac{p}{q} : p, q \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\}.$$

Theorem (Somorjai-Bak-Newman). $R(\Lambda) \cap C[0,1]$ *is always dense in* $C[0,1]$.

So division has extra utility. Can multiplication have this extra utility? In [*, p. 50] Newman writes "Thus we have the very sane, if very prosaic, question:

P(10.6) Are the functions

$$\left(\sum a_i x^{i^2}\right)\left(\sum a_j x^{j^2}\right)$$

dense in C[0,1]?"

2. NEWMAN'S PRODUCT PROBLEM (1978)

Let

$$\Lambda := (\lambda_j)_{j=0}^{\infty}$$

be a sequence of distinct nonnegative real numbers with $\lambda_0 = 0$. Suppose

$$M^k(\Lambda) := \left\{ \prod_{j=1}^{k} p_j : p_j \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\}.$$

Suppose $k \geq 2$ and

$$(*) \qquad \sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty.$$

Can $M^k(\Lambda)$ be dense in $C[0,1]$?

Theorem 2.1. *If (*) holds, $k \geq 2$, and $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $M^k(\Lambda)$ is not dense in $C(A)$.*

Remark 2.2. *$M^k(\Lambda)$ is contained (not equal to)*

$$\operatorname{span}\{x^{\lambda_{j_1}+\lambda_{j_2}+\cdots+\lambda_{j_k}} : \lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_k} \in \Lambda\}.$$

Example 2.3. *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be defined by*

$$\lambda_j := \begin{cases} 0, & j = 0 \\ 2^{j-1}, & j = 1, 2, \ldots . \end{cases}$$

Then

$$\sum_{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_k} \in \Lambda} \frac{1}{\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_k}} < \infty$$

so it follows from Müntz's Theorem that $M^k(\Lambda)$ is not dense in $C[0,1]$.

Example 2.4. *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be defined by*

$$\lambda_j := j^2.$$

Then

$$\begin{aligned} M^4(\Lambda) &\subset \operatorname{span}\{x^{k^2+l^2+m^2+n^2} : k, l, m, n \in \mathbb{N}\} \\ &= \operatorname{span}\{x^n : n \in \mathbb{N}\}. \end{aligned}$$

So in this case the non-denseness of $M^4(\Lambda)$ is not obvious at all.

Theorem 2.5 (Remez-Type Inequality).
For every sequence

$$\Lambda := (\lambda_j)_{j=0}^{\infty} \qquad \textit{with} \qquad \sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty$$

there is a constant c depending only on Λ and s (and not on A or the number of terms in p) so that

$$\|p\|_{[0,\inf A]} \le c\,\|p\|_A$$

for every

$$p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

and for every $A \subset [0,1]$ of Lebesgue measure at least $s \in (0,1)$.

3. Full Müntz Theorem in $L_p[0,1]$ for $p \in (0,\infty)$

Theorem 3.1. *Let* $p \in (0,\infty)$. *Suppose* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct real numbers greater than* $-(1/p)$. *Then*

$$\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $L_p[0,1]$ *if and only if*

$$(*) \qquad \sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty\,.$$

Moreover, if (*) does not hold, then every function from the $L_p[0,1]$ *closure of*

$$\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

can be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$$

restricted to $(0,1)$.

4. FULL CLARKSON-ERDŐS-SCHWARTZ THEOREM IN $C[0,1]$

The right result when $p = \infty$ is proved in a separate paper.

Theorem 4.1. *Let* $(\lambda_j)_{j=1}^{\infty}$ *be a sequence of distinct positive numbers. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *if and only if*

$$(*) \qquad \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty \,.$$

Moreover, if (*) does not hold then every function from the $C[0,1]$ *closure of* $H(\Lambda)$ *can be represented as an analytic function on*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$$

restricted to $(0,1)$*.*

This result improves an earlier stating that if (*) does not hold then then every function from the $C[0,1]$ closure of $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is in $C^\infty(0,1)$.