Ramsey Theory: The New

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“Density and Arithmetic Progressions with Odd Differences” (H. Ardal, T. Brown, V. Jungic, H. Zhai)

In a 2-colouring, if the density of a colour is t on each modulo 4 congruence class, then there is an $AP(4)$ in that colour having odd common difference.

(1/2, 3/4)

Definitions

- *k*-colouring

  A *k*-colouring is a distinct labeling of each positive integer so that the natural numbers are partitioned into *k* cells.

- Arithmetic Progression

  A sequence of numbers where the difference between successive terms is some constant.

Theorems

- *Van der Waerden's Theorem*

  If the natural numbers are partitioned into k cells, then there exists an arbitrarily long arithmetic progression in one of the cells.

- *Szemeredi's Theorem*

  If *A* is a set of positive integers with positive upper density, then *A* contains arbitrarily long APs.

  (long APs in the "most frequently occurring" colour)

Restrict to Odd Difference

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1 2 3 4 5 6 7 8 9 10 11 12

No monochromatic AP(2) with odd difference

Question

Is there a condition on the colouring of $\mathbb{N}$ that will guarantee the existence of an *AP(4)* with odd difference?

Answer

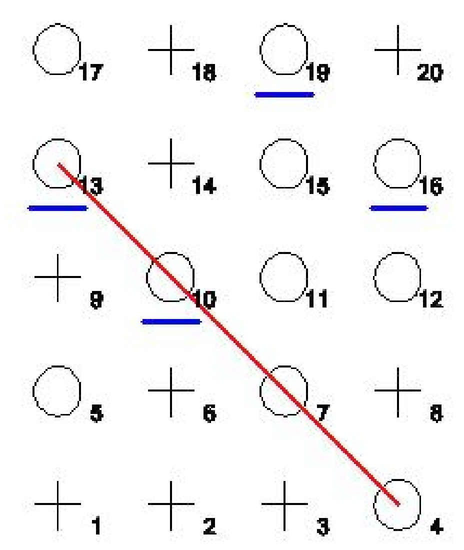
Density on each modulo 4 congruence class

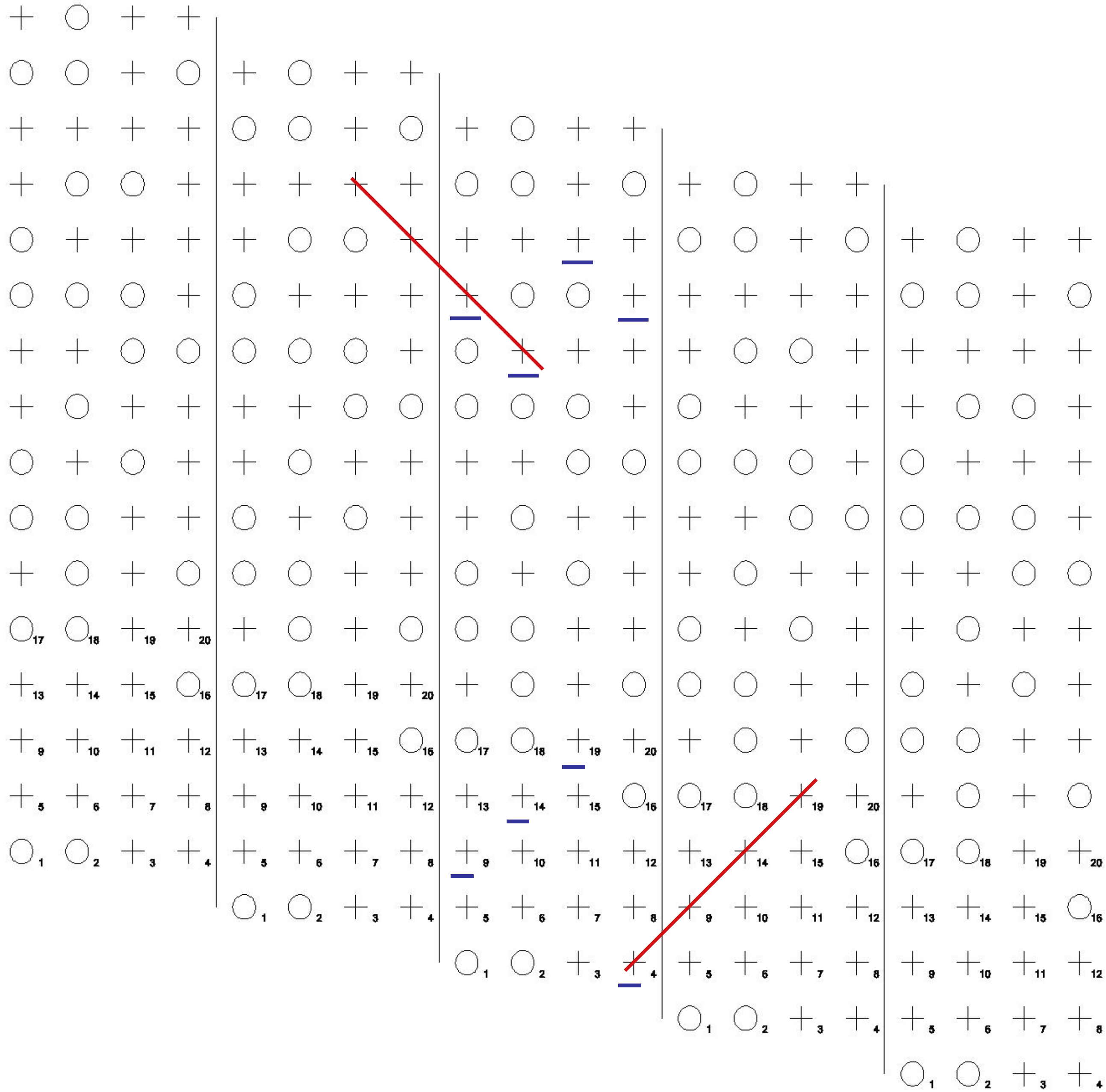
Geometric Interpretation

Each column is a modulo 4 congruence class.

Each *AP* of odd difference becomes

- a straight line in the plane, or
- one that is curved around an imaginary cylinder





In the infinite case, there exists a construction of a two colouring with the upper density of one of the colours equal to 1 on each modulo 4 congruence classes that avoids a monochromatic *AP(3)* having odd common difference.

Thus, the size of the upper density of a colour on modulo 4 congruence classes does not guarantee the existence of a monochromatic *AP(3)* having odd difference.

In the finite case, a density $t \in (1/2, 3/4]$ of one of the colours on each of the modulo 4 congruence classes guarantees a monochromatic AP(4) having odd difference.