

The background features a complex geometric design with orange lines forming various shapes and patterns. There are several horizontal and vertical lines, some forming a grid-like structure. There are also several clusters of small orange dots, some arranged in rows and others in more irregular patterns. The overall aesthetic is modern and mathematical.

# Ramsey Theory: The New

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“Density and Arithmetic Progressions with  
Odd Differences” (H. Ardal, T. Brown, V.  
Jungic, H. Zhai)

In a 2-colouring, if the density of a  
colour is  $t$  on each modulo 4  
congruence class, then there is an  
 $AP(4)$  in that colour having odd  
common difference.

$(1/2, 3/4)$

# Definitions

- $k$ -colouring

A  $k$ -colouring is a distinct labeling of each positive integer so that the natural numbers are partitioned into  $k$  cells.

- Arithmetic Progression

A sequence of numbers where the difference between successive terms is some constant.

# Theorems

- *Van der Waerden's Theorem*

If the natural numbers are partitioned into  $k$  cells, then there exists an arbitrarily long arithmetic progression in one of the cells.

- *Szemerédi's Theorem*

If  $A$  is a set of positive integers with positive upper density, then  $A$  contains arbitrarily long APs.

(long APs in the “most frequently occurring” colour)

# Restrict to Odd Difference

■ ○ ■ ○ ■ ○ ■ ○ ■ ○ ■ ○ ...

1 2 3 4 5 6 7 8 9 10 11 12

No monochromatic AP(2) with odd difference

## *Question*

Is there a condition on the colouring of  $\mathbb{N}$  that will guarantee the existence of an *AP*(4) with odd difference?

## *Answer*

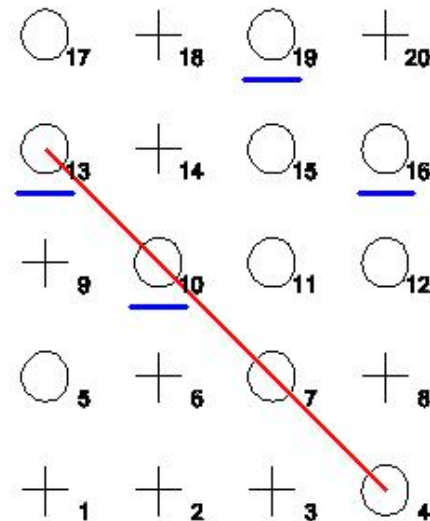
Density on each modulo 4 congruence class

# Geometric Interpretation

Each column is a modulo 4 congruence class.

Each *AP* of odd difference becomes

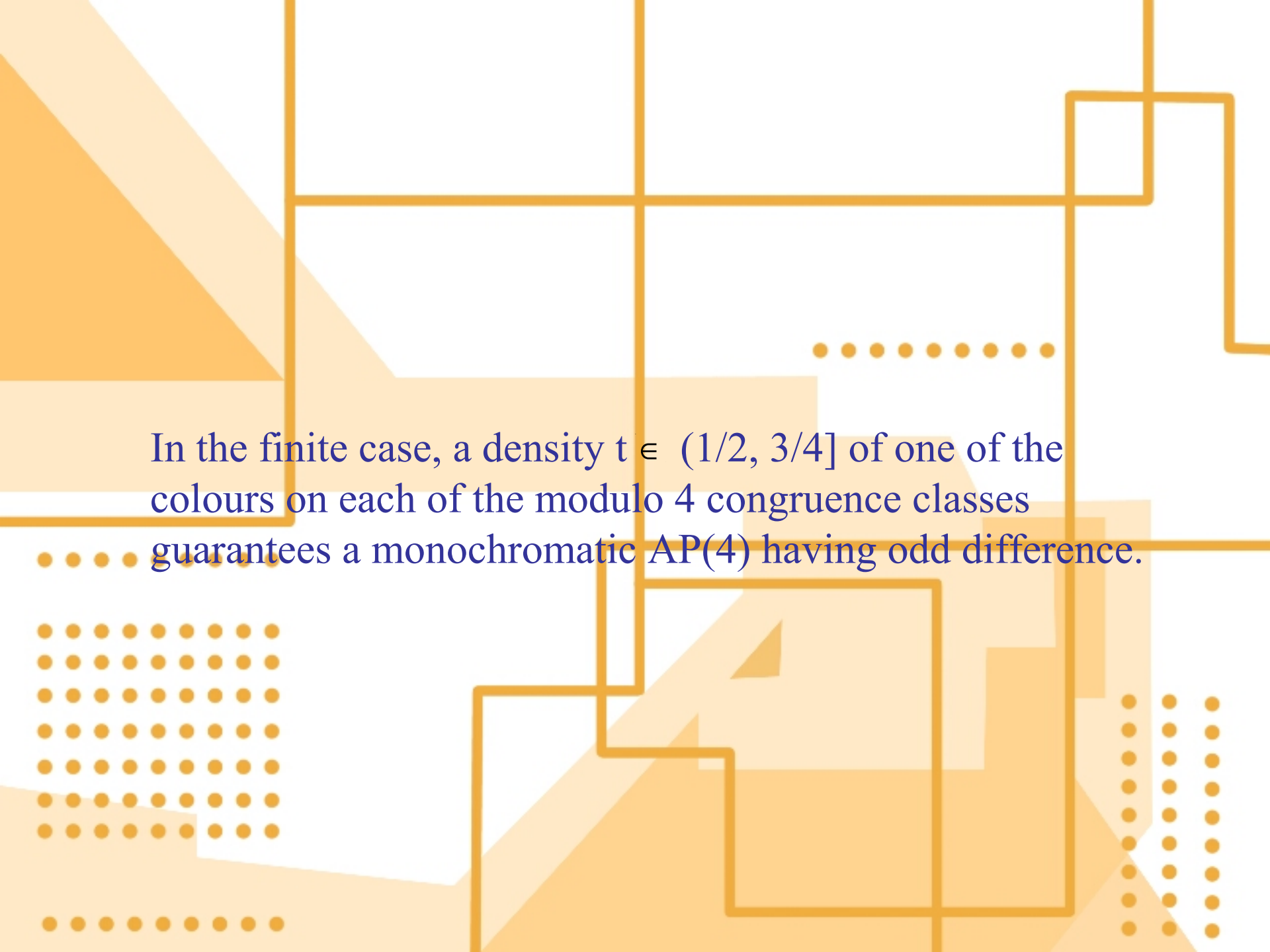
- a straight line in the plane, or
- one that is curved around an imaginary cylinder





In the infinite case, there exists a construction of a two colouring with the upper density of one of the colours equal to 1 on each modulo 4 congruence classes that avoids a monochromatic  $AP(3)$  having odd common difference.

Thus, the size of the upper density of a colour on modulo 4 congruence classes does not guarantee the existence of a monochromatic  $AP(3)$  having odd difference.

The background features a complex geometric pattern of orange lines and dots. A prominent feature is a large, irregular shape on the left side, composed of several overlapping triangles and rectangles. A series of orange dots is arranged in a grid-like pattern on the left side, with a horizontal row of dots above the main text and a vertical column of dots to the right. The overall aesthetic is modern and mathematical.

In the finite case, a density  $t \in (1/2, 3/4]$  of one of the colours on each of the modulo 4 congruence classes guarantees a monochromatic AP(4) having odd difference.