Introduction to crossing number of graphs

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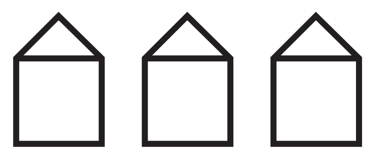

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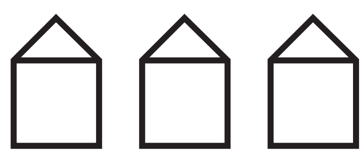

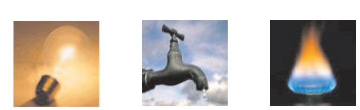

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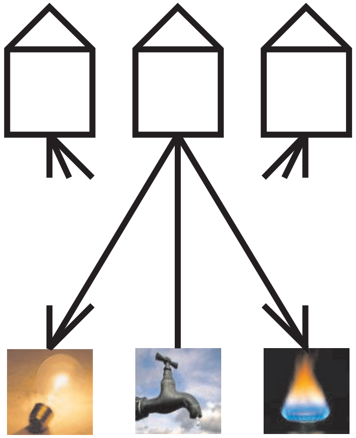

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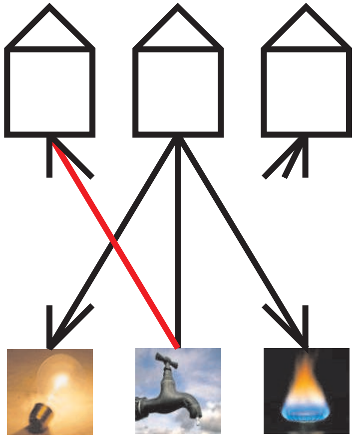

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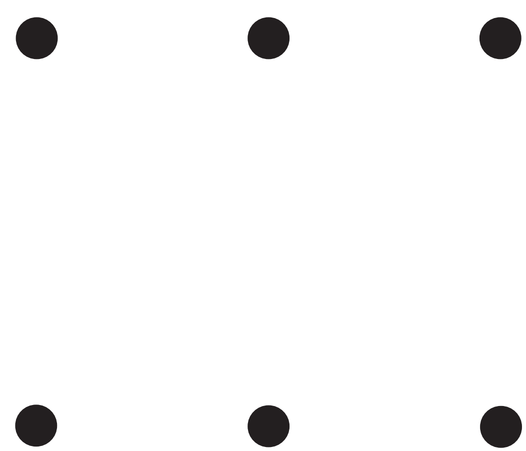
- Anonymous:
  - "I remember I succeeded a couple of years ago, I just can't remember how I did it!"

Model of the problem: a graph $K_{3,3}$.

- Houses, utilities: 6 **vertices**.



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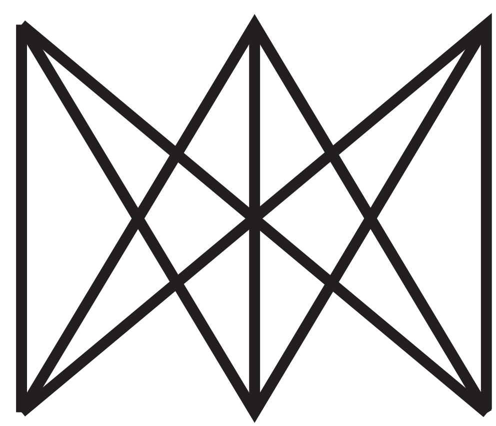

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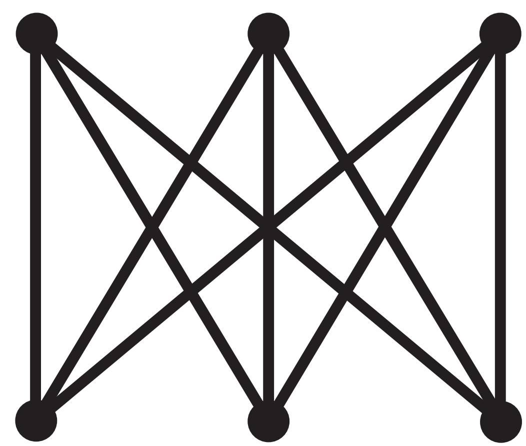

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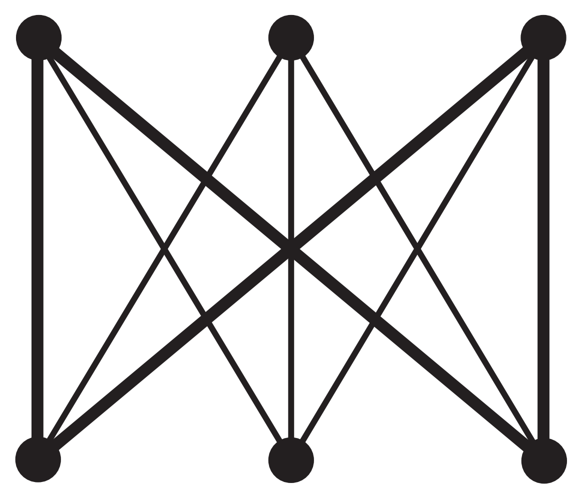
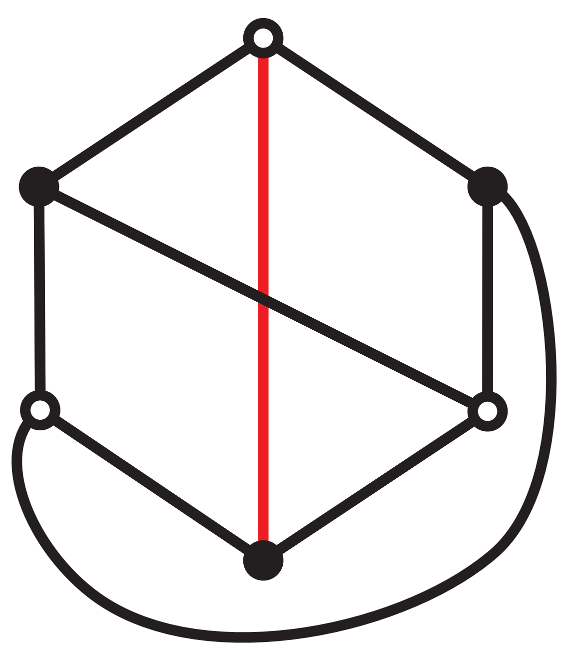
- Shortest roundtrip: shortest **cycle** of length 4.



Model of a solution: an embedding.

- Nonintersecting **points** and **curves** in the plane.



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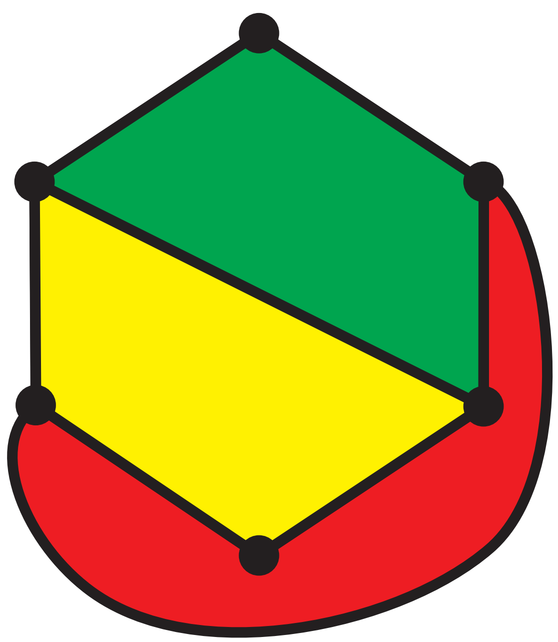
- Roundtrips with nothing in the interior: **faces** of the embedding.



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- $18 \geq 20$.
- Contradiction:
  an embedding of $K_{3,3}$ does not exist.

Examples of graphs

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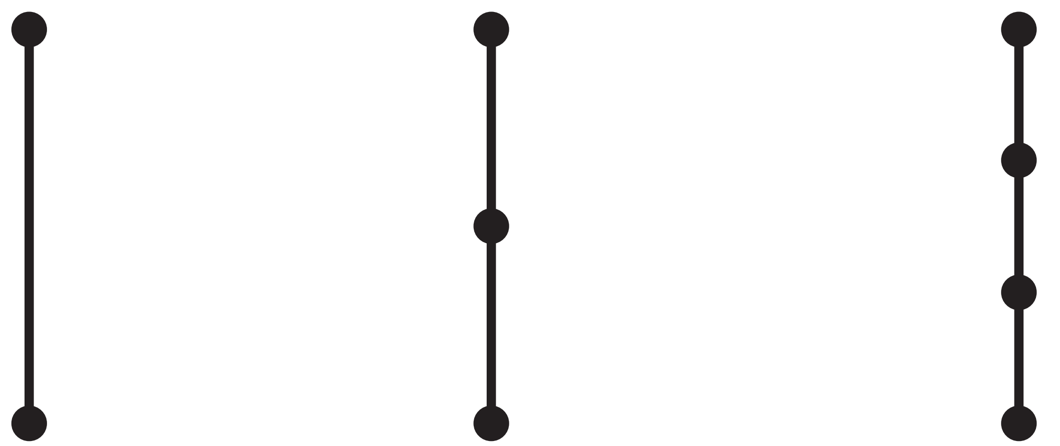

P_1 P_2 P_3

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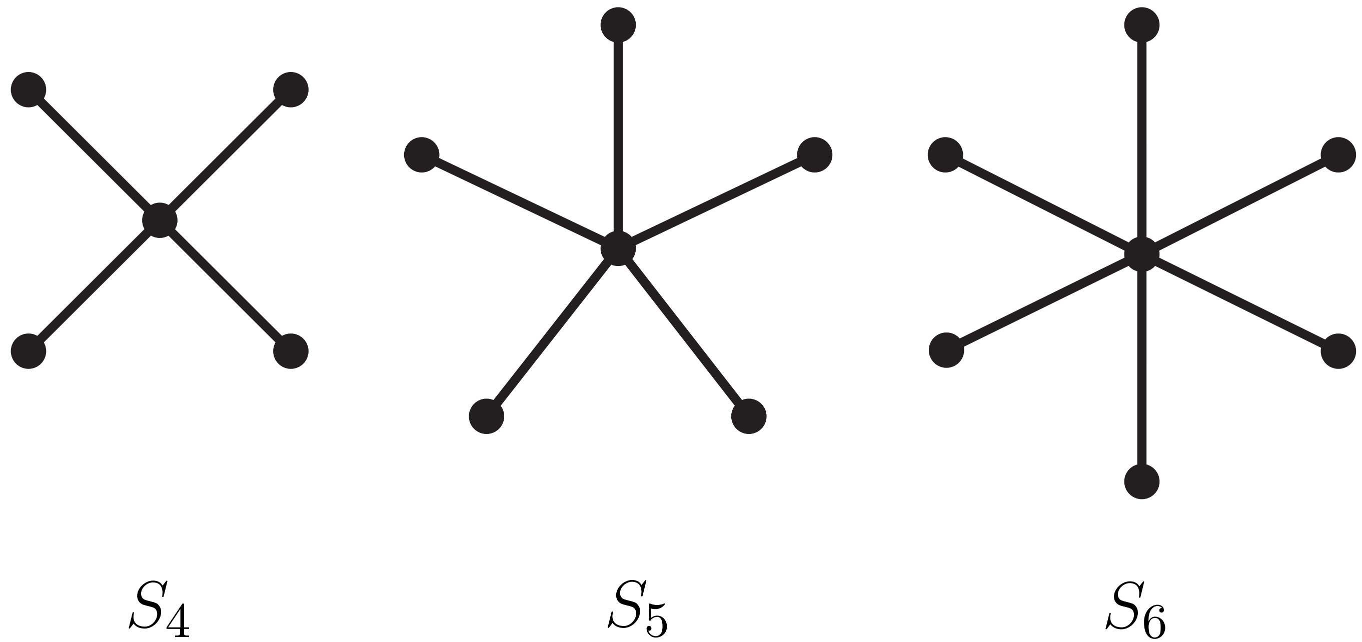

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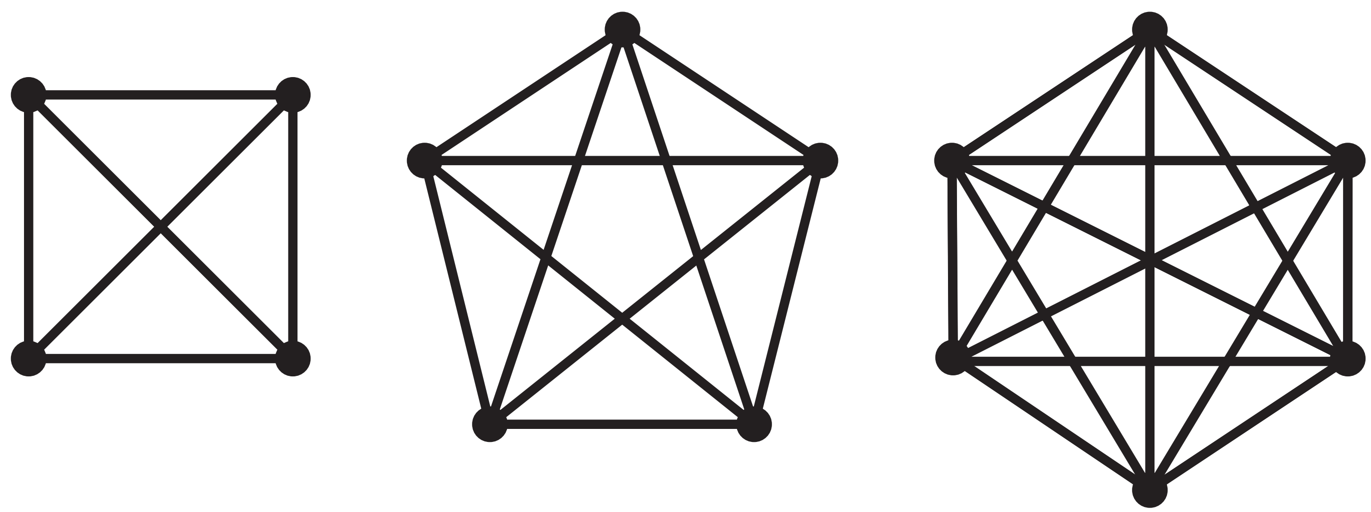
- **Complete graph** K_n on n vertices.



K_4 K_5 K_6

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  - no three edges cross at the same point.

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- **Crossing number** of a drawing $\mathrm{cr}(D)$: number of crossings between edges in D.
- **Crossing number** of a graph $\mathrm{cr}(G)$: smallest crossing number of some drawing of G.
- Example: $\mathrm{cr}(K_{3,3}) = 1$.

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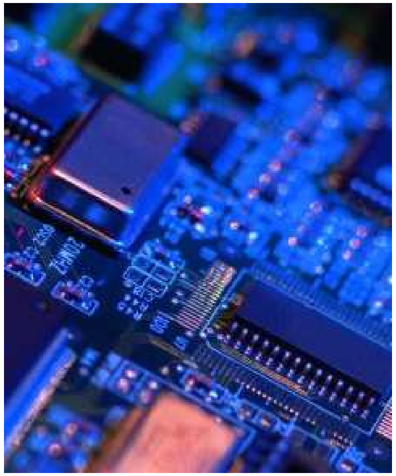

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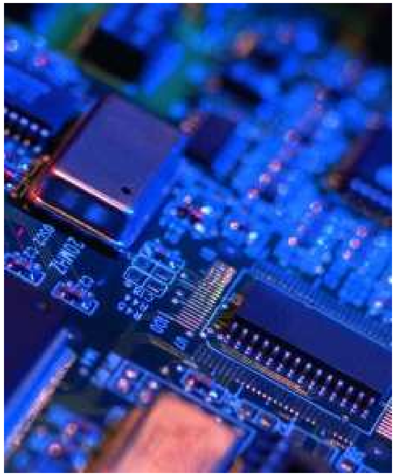

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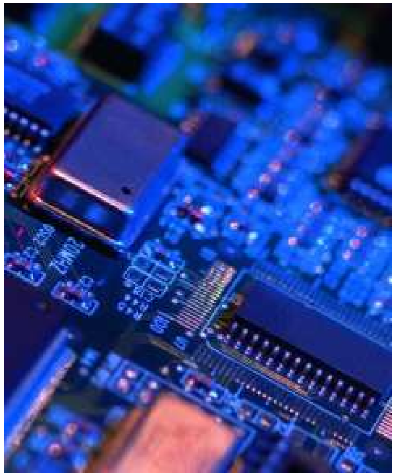

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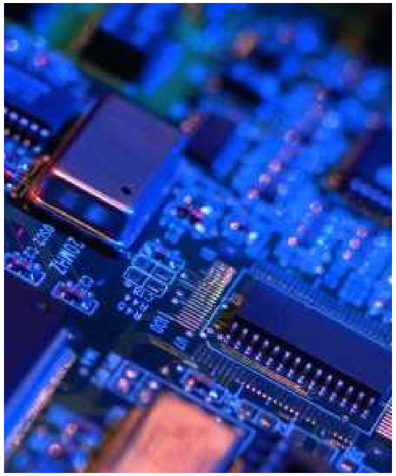

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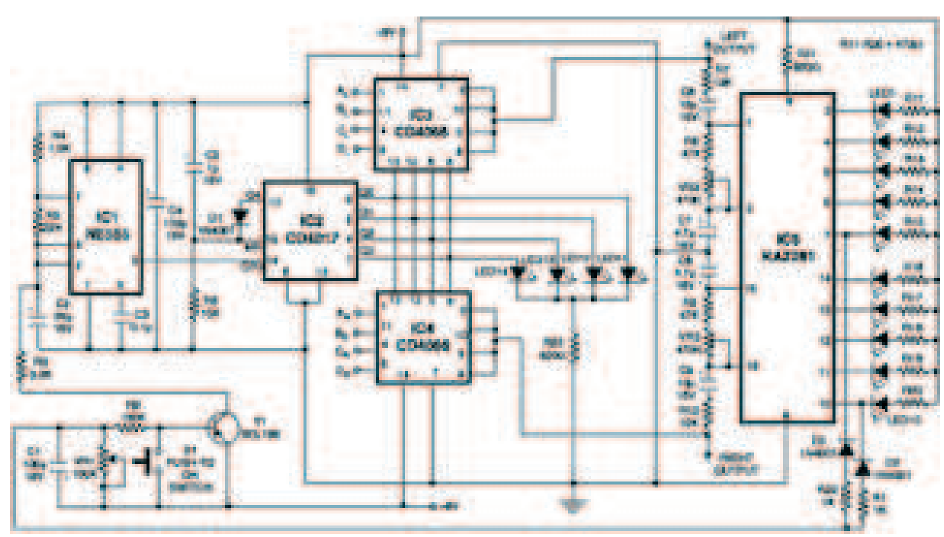

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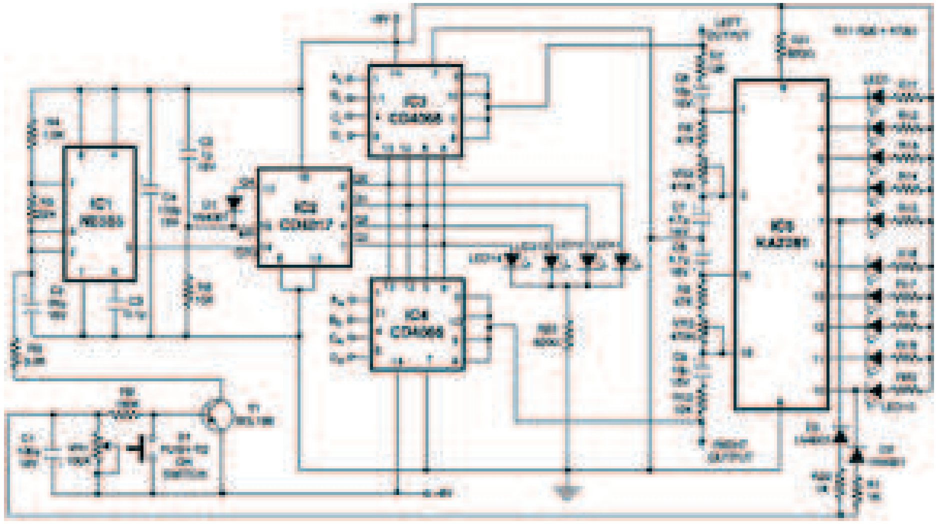

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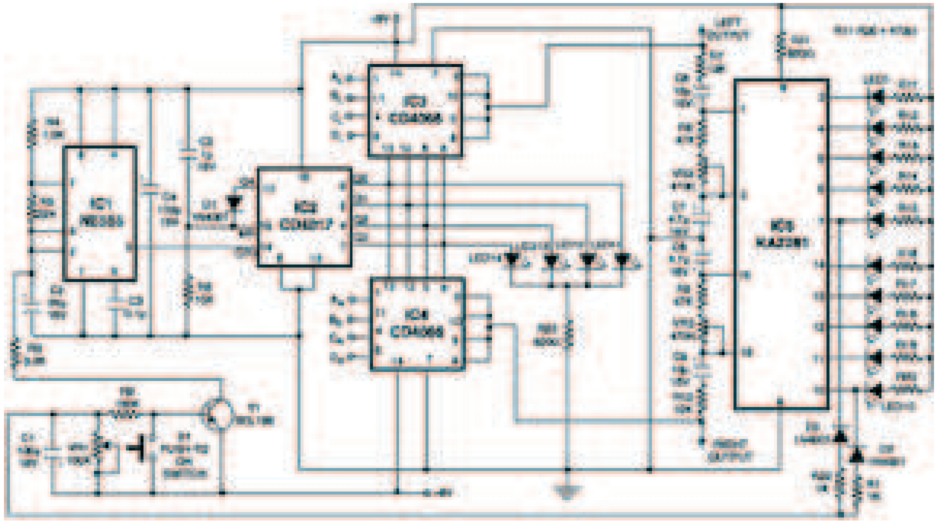
  - Quickly computing n-variable transitive function: many crossings in the chip.



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- Székely (1997):
  - For any n points in the plane, there is one among them with at least $c \cdot n^{4/5}$ distinct distances to the other $n-1$ points.

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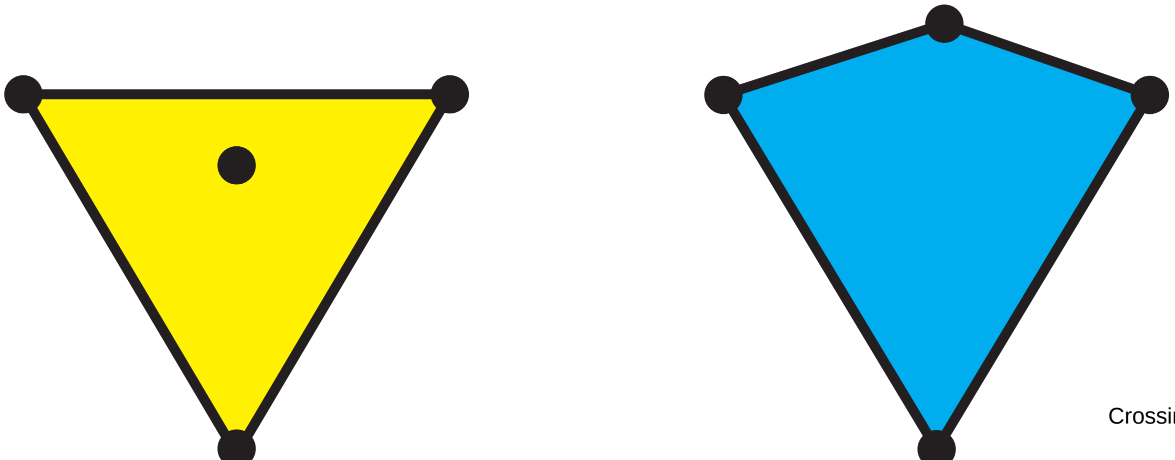

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- Theorem (Balogh, Salazar, 2004): $c^* \geq 0.37553$.

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  - Lower bounds: various specific approaches.

A general lower bound

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  - $p^4\,\mathrm{cr}(G) \geq p^2 m - 3pn$.

Zip product

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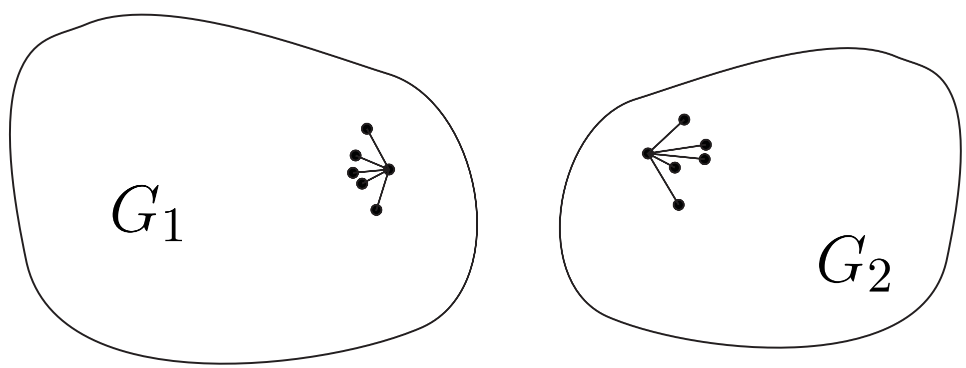

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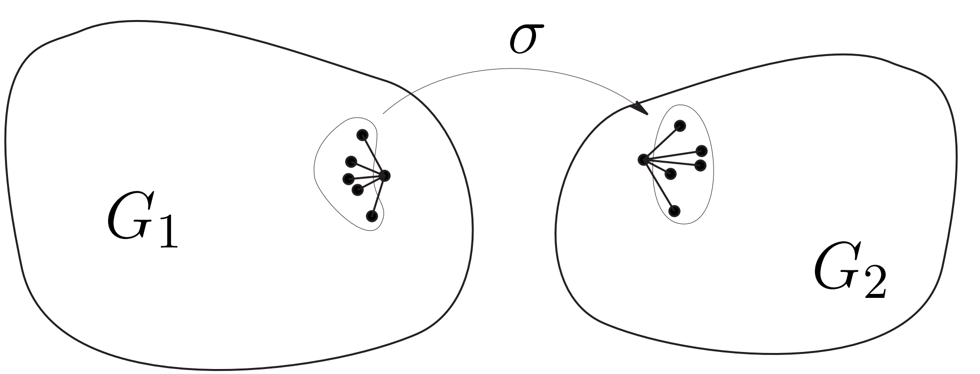

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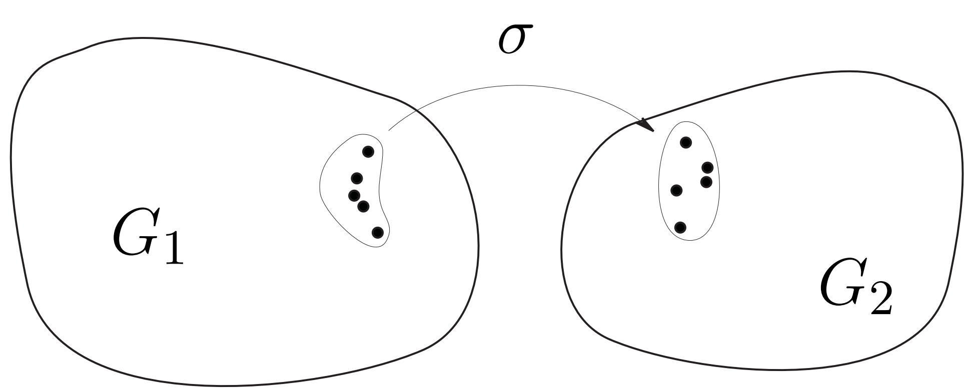

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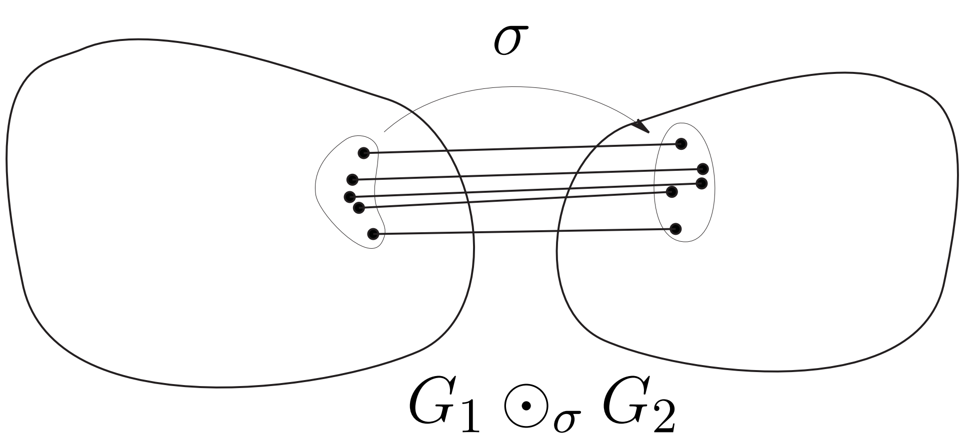

The key Lemma of the Zip product

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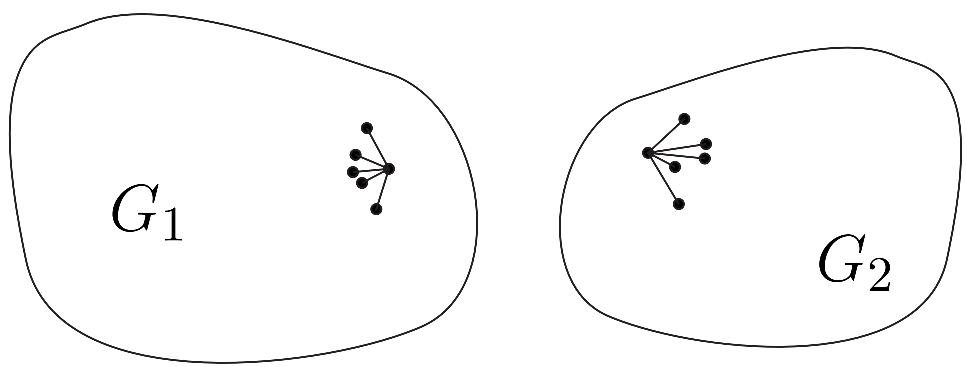

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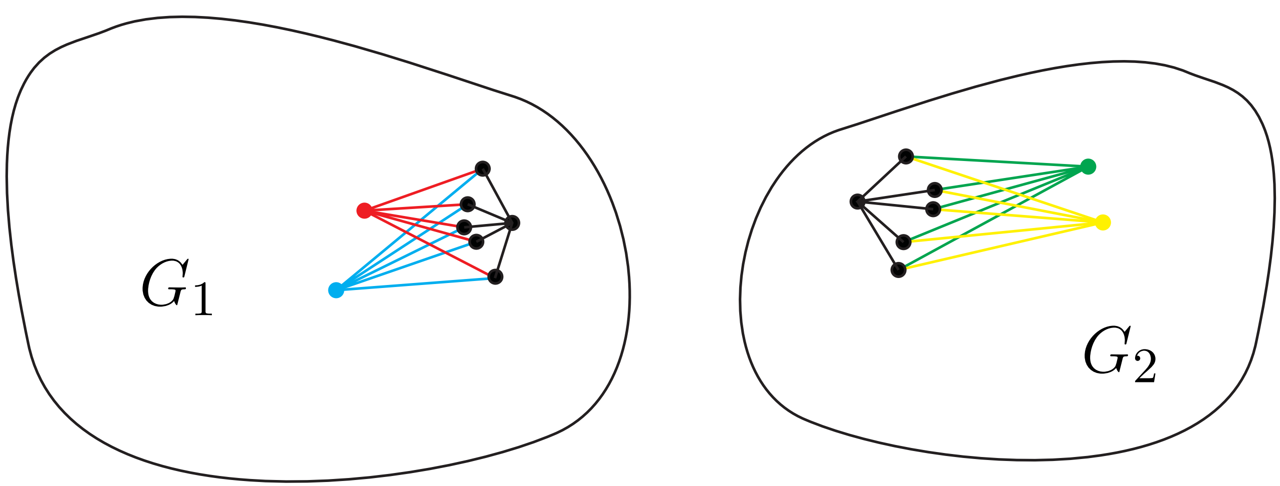

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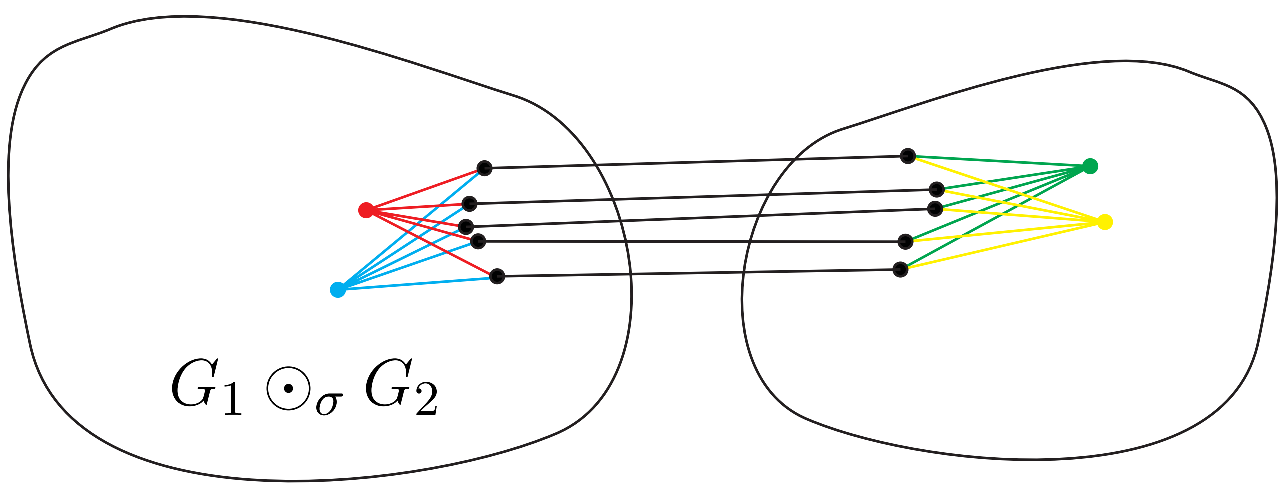

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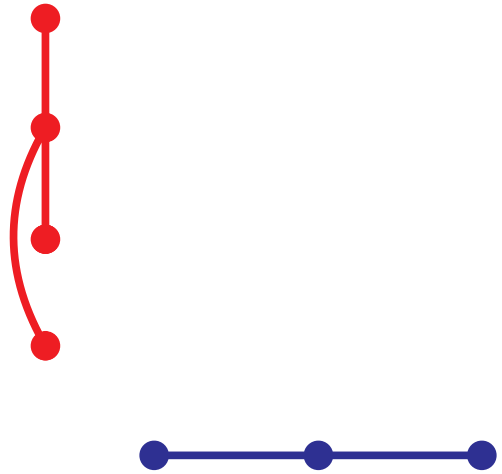

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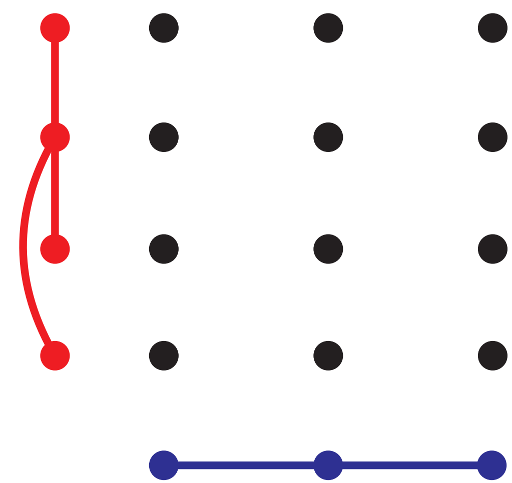

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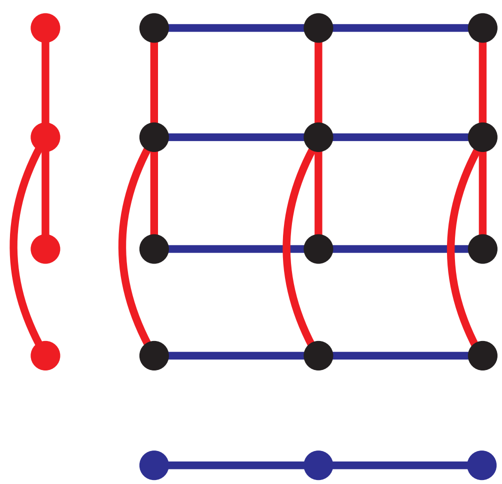
  - $(v_1, v_2)(v_1', v_2')$ is an edge of G,
    if $v_1 v_1'$ or $v_2 v_2'$ is an edge.



Crossing number of $S_m \Box P_n$

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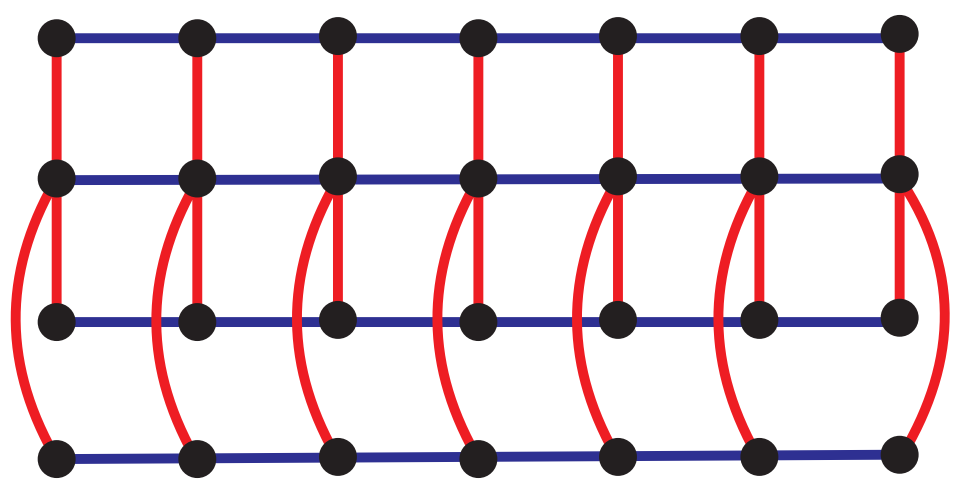

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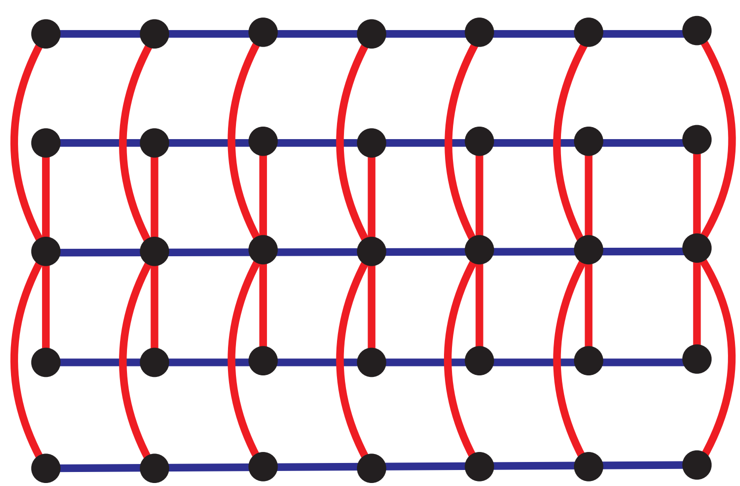
- Theorem (Klešč, 1991): $\mathrm{cr}(S_4 \square P_n) = 2(n-1)$.



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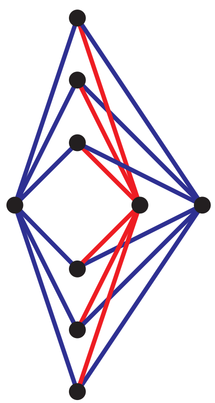

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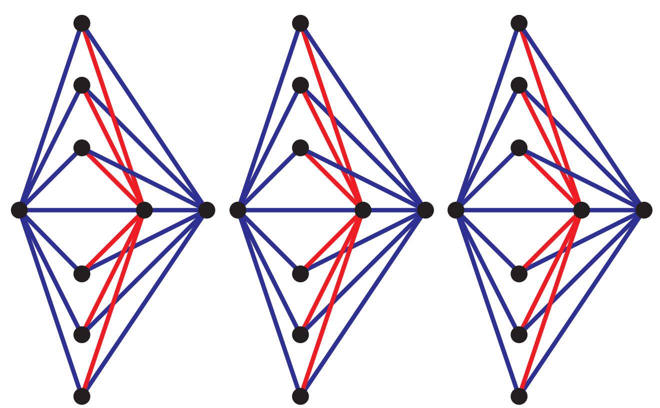

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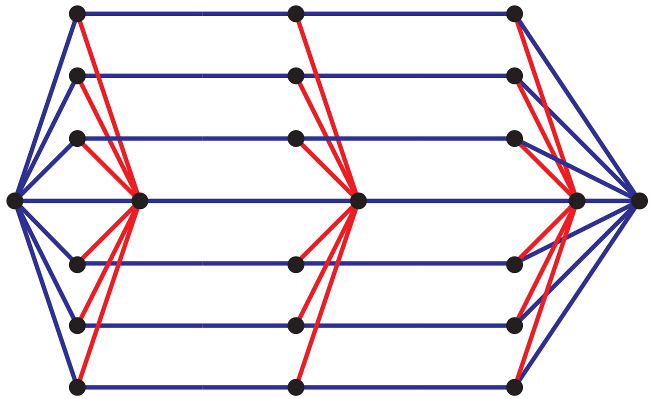

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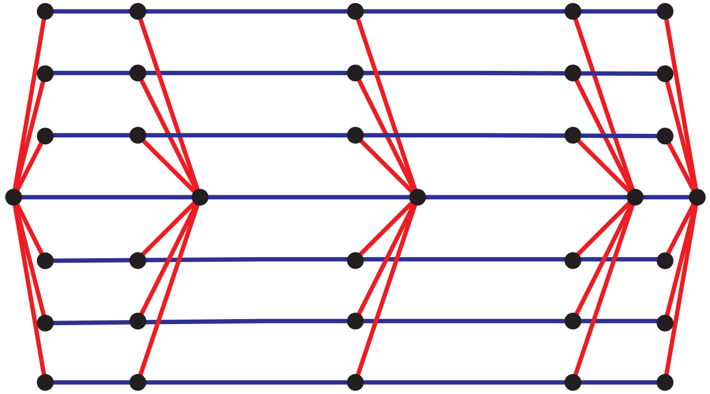

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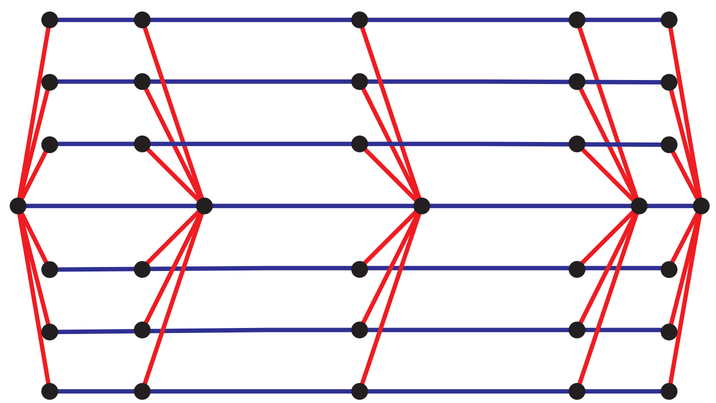
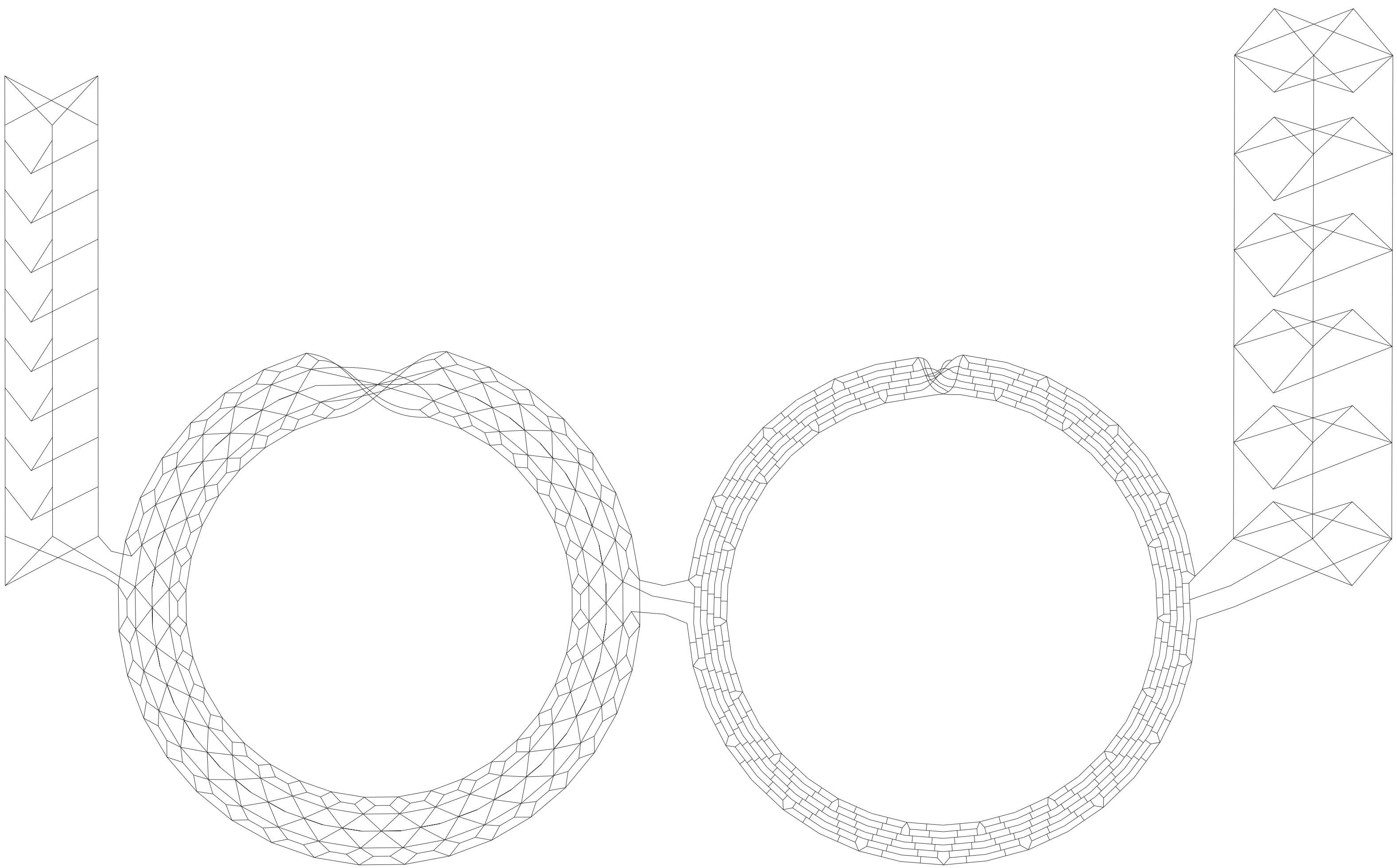
  - By the lemma: $\mathrm{cr}(S_m \Box P_n) \geq (n-1) \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$.



An “spectacle graph” graph

$G \in \Gamma(7, 37, 8, 0, 34, 10, 6)$



- There exists a convex continuous function $f : (3, 6) \to \mathbb{R}^+$ such that for $r \in (3, 6) \cap \mathbb{Q}$ and $k \geq f(r)$, there exists an infinite family of 3-connected simple crossing critical graphs with average degree r and crossing number k.

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- For any closed interval $I = [r_1, r_2] \subseteq (3,6)$ set $N_I = \max_i f(r_i)$.

  Then for every $r \in I \cap \mathbb{Q}$ and every $k \geq N_I$ there exists an infinite family as above.

A constructive version

$$f(r) = 240 + \frac{512}{(6-r)^2} + \frac{224}{6-r} + \frac{25}{16(r-3)^2} + \frac{40}{r-3}$$