Shape Restricted Estimation of Dark Matter Distributions In Dwarf Spheroidal Galaxies

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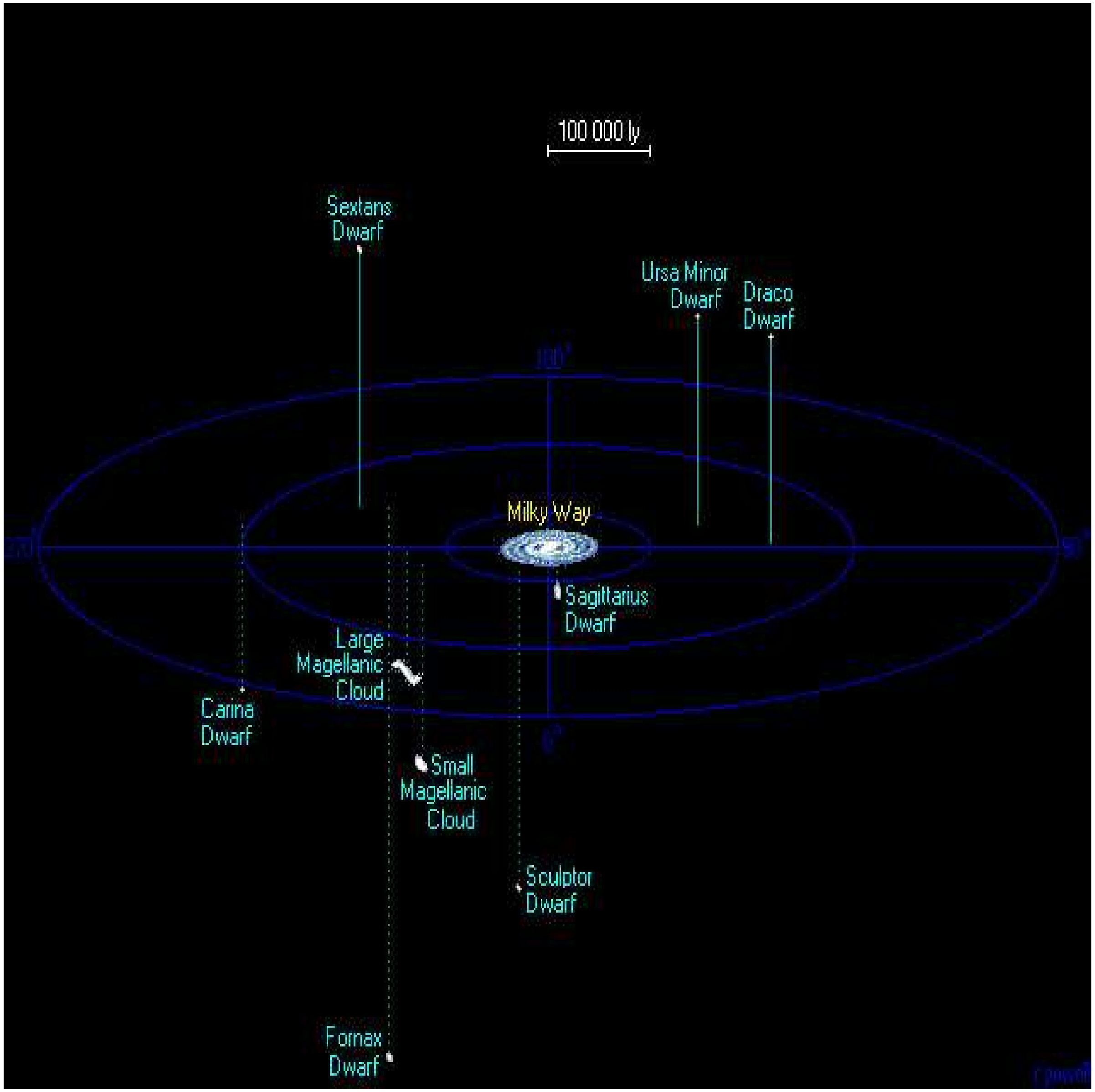
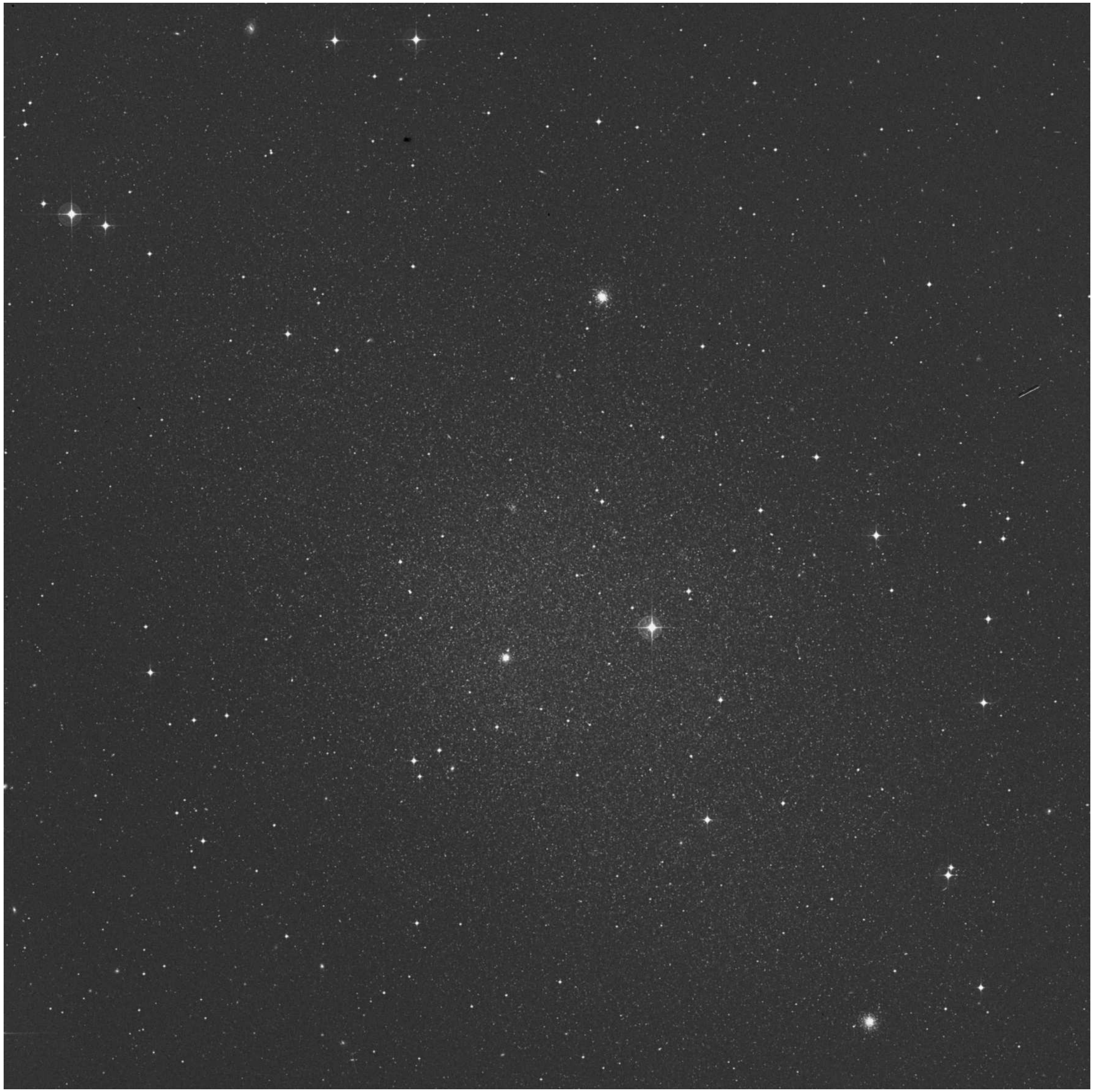
Outline

- The Astronomical Problem: Distribution of DM
    - -The Dwarf Spheroidals
    - -Dark Matter
- Statistical Issues
    - -Inverse Problem
    - -Missing Data
    - -Shape Restrctions
- Bean Counting
- Complications and Extensions

The Dwarf Spheroidals



Fornax



Dark Matter

Existence: If all matter were visible, then stars wouldn't be where they are or move as they do!

What is it?

- Don't Know (Wimps, Machos, $\cdots$).
- Exclusions–e.g. dead stars.

Where it is?

- This is our objective.
- May shed light on what it is.

The Data

- Projected Positions (Easy).
- Line of Sight Velocities.(Harder).
  -From Doppler Effect
  -Need time to get enough light.

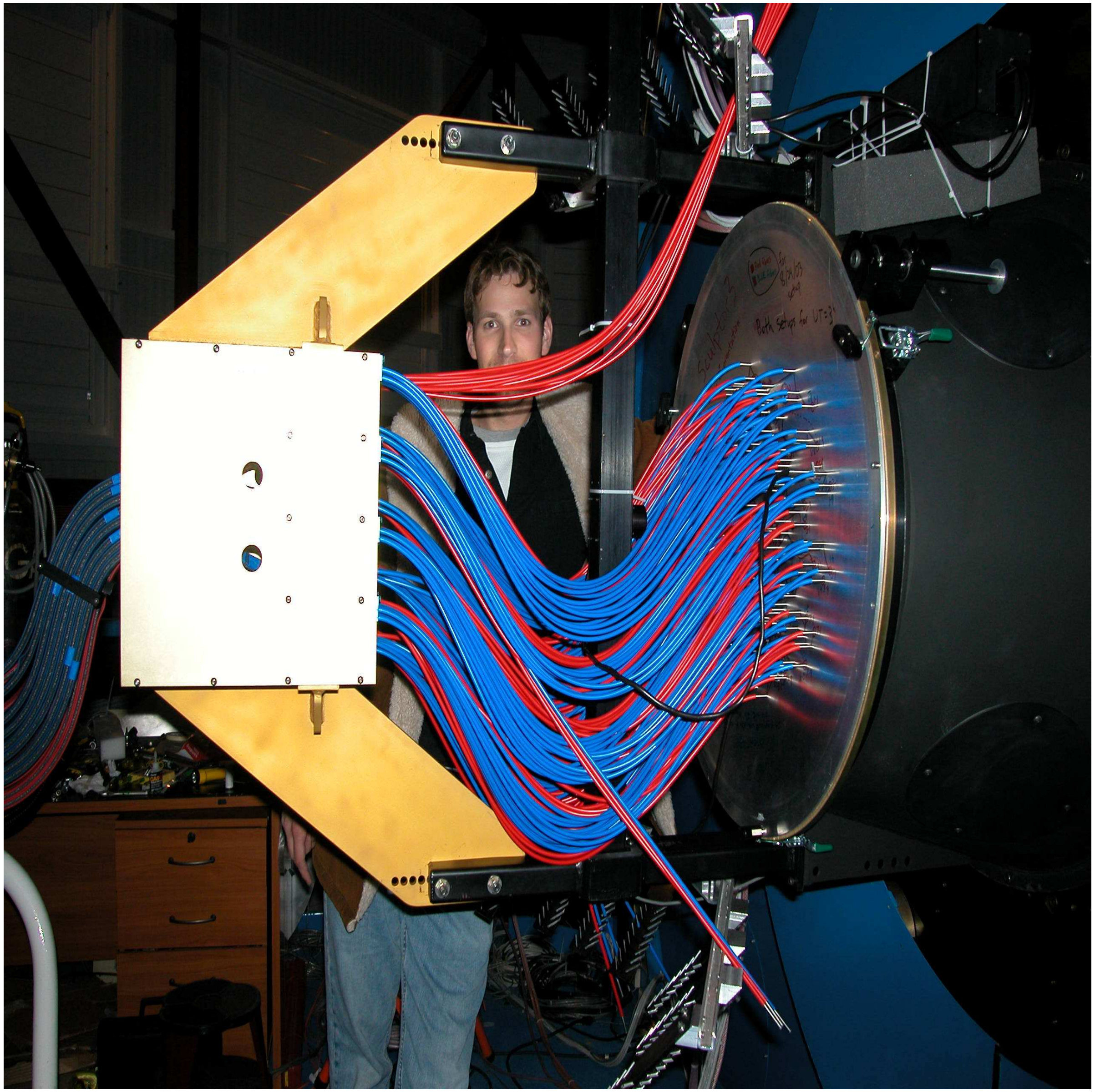
Statistically

- Inverse Problem.
- Missing Data.
- Non-transparent shape restrictions.

What' ν?

- The Fibers: Light from many stars at once.
- Non-parametric analysis.
  -First study used four stars: later ones up to 50.
  -Parametric models

The Fibers



Notation

Positions and Velocities: For a star in a dSph, let

$$\boldsymbol{X} = (X_1, X_2, X_3) \qquad \text{and} \qquad \boldsymbol{V} = (V_1, V_2, V_3)$$

Density: Regard these as random vectors and suppose

$$P\big[\boldsymbol{x} \le \boldsymbol{X} \le \boldsymbol{x} + \boldsymbol{dx},\ \boldsymbol{v} \le \boldsymbol{V} \le \boldsymbol{v} + \boldsymbol{dv}\big] = f(r^2, v)\boldsymbol{dx}\boldsymbol{dv},$$

where

$$r^2 = x_1^2 + x_2^2 + x_3^2 \qquad \text{and} \qquad v^2 = v_1^2 + v_2^2 + v_3^2$$

Normalizations: Here $\mathbf{0} = (0, 0, 0)$ is the center of the galaxy, and

$$\int \boldsymbol{v} f(r^2, v)\boldsymbol{dx}\boldsymbol{dv} = \mathbf{0}.$$

Mass Distribution

Let $\rho(r)$ be the mass density and

$$M(r) = 4\pi \int_0^r t^2 \rho(t) dt.$$

Jean's Equation: Assuming equilibrium and isotropy

$$M(r) = -\frac{r^2 \mu(r)}{G f(r^2)} \frac{d}{dr} \log\left[\mu(r)\right],$$

where

$$f(r^2) = \int f(r^2, v) \boldsymbol{dv},$$

$$\mu(r) = \frac{1}{3} \int v^2 f(r^2, v) \boldsymbol{dv},$$

Velocity Dispersions

Observe that

$$E[V^2|\boldsymbol{X} = \boldsymbol{x}] = \frac{3\mu(r)}{f(r^2)}.$$

Here:

- $f(r^2)$ is just the marginal density of $\boldsymbol{X}$;
- $\sqrt{[E(V^2|\boldsymbol{X} = \boldsymbol{x}]}$ is called the *velocity dispersion*.

Two Related Problems

- Estimate f: Easier–census
- Estimate μ: Harder–sample.

Plummer's (1911) Model

$\boldsymbol{X} = (X_1, X_2, X_3)$ and $\boldsymbol{V} = (V_1, V_2, V_3)$ have joint density

$$\frac{c_0}{b^5\sqrt{u}}\left[\frac{b}{\sqrt{1+\frac{1}{3}r^2}} - \frac{1}{2}v^2\right]_+^{\frac{7}{2}},$$

where $x_+ = \max[0, x]$, c_0 is a normalizing constant, and b is a parameter that is related to the velocity dispersion through

$$E[V^2|R=r] = \frac{b}{6\sqrt{1+\frac{1}{3}r^2}}.$$

Missing Data

Observe:

- Line of sight velocities.
- Projected positions on the orthogonal plane.

With a proper choice of coordinates, we observe X_1, X_2, and V_3. Here

$$\mu(r) = \int_{\mathbb{R}^3} v_3^2 f(r^2, v) d\boldsymbol{v},$$

but the missing X_3 causes more serious problems, known as Wicksell's Problem.

Refs: Wicksell (1925, *Biometrika*).

Groeneboom and Jongbloed (1995, *Ann. Statist.*).

179 Fornax Stars

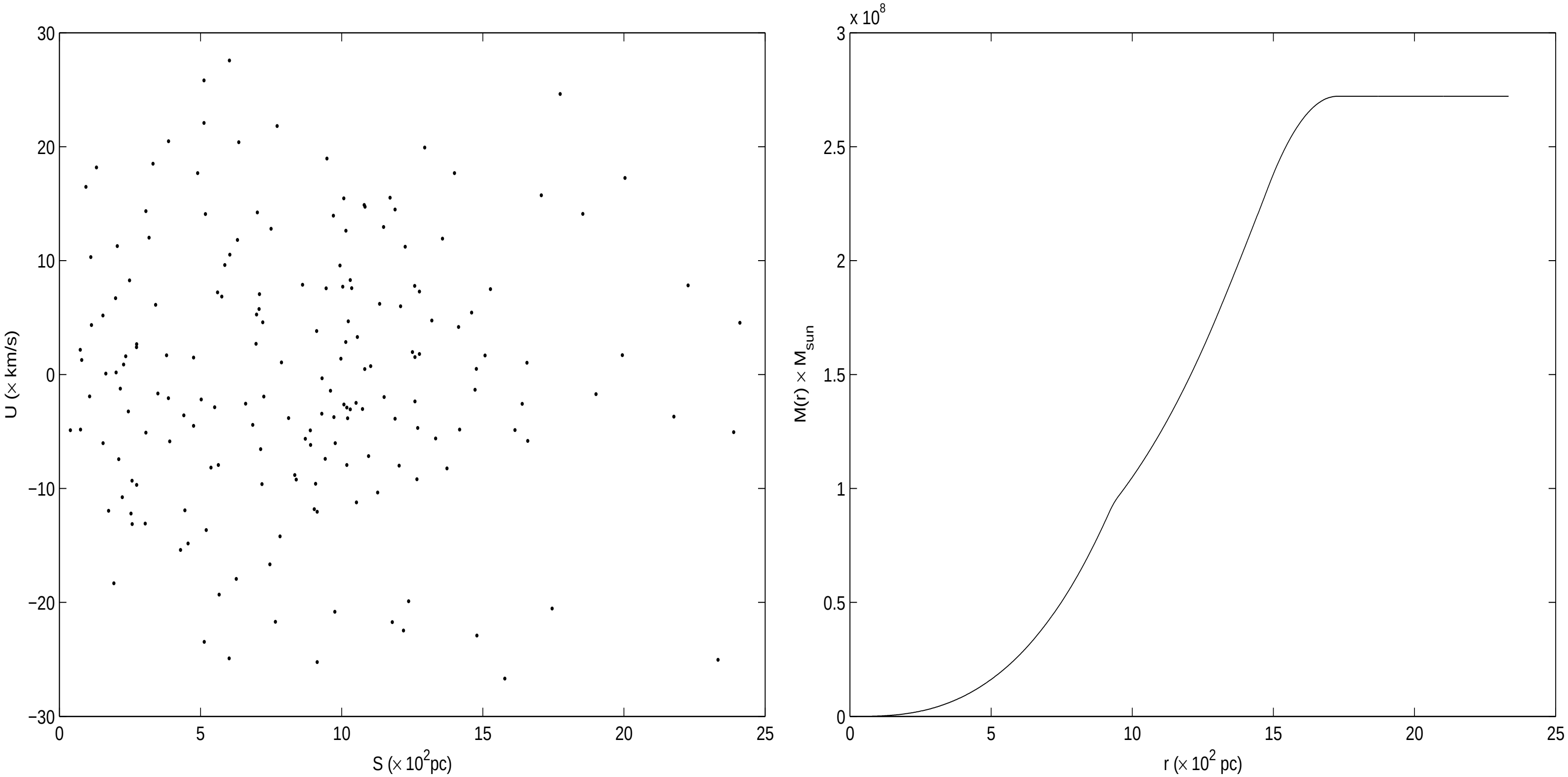


Figure 1: $S = \sqrt{X_1^2 + X_2^2}$ is distance; $U = V_3$ is velocity

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Estimating f

Large Data Sets: e.g for Fornax

Star Counts From Fornax

r	d	N
1.05	29.33	97
2.10	30.57	304
3.15	31.31	519
	$\dots$	
109.11	1.34	0

Ref: Irwin and Hatzidimitriou (1995, A.J.)

Estimating f: Continued

Squared Radii: Let

$$Y = X_1^2 + X_2^2,$$
$$Z = X_1^2 + X_2^2 + X_3^2.$$

Estimation

- Relate the distribution of Y to f.
- Estimate the distribution of Y.
- Solve for f.
- Require f to be decreasing (Optional).

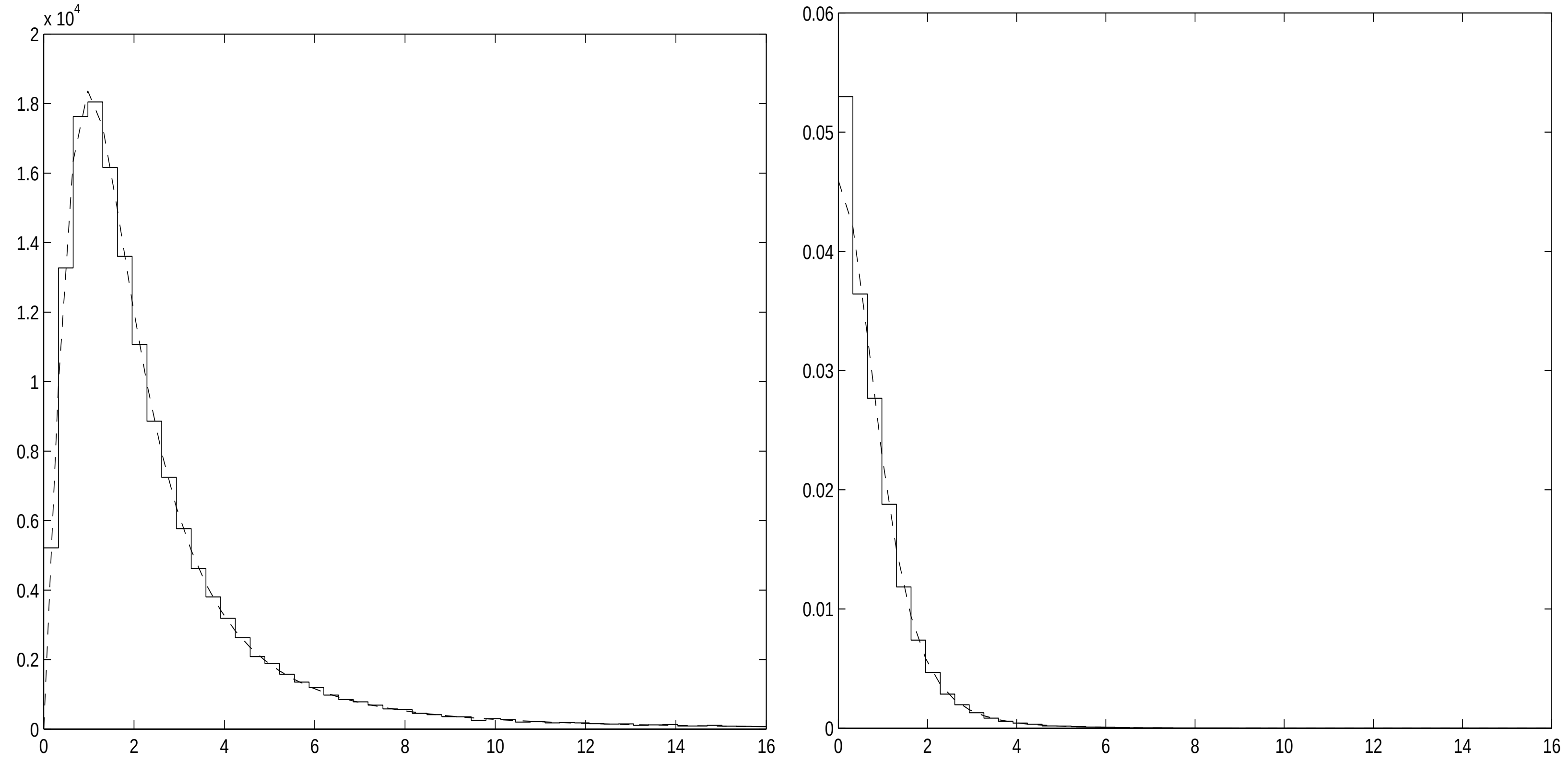

Figure 2: Counts and $\hat{f}(r^2)$ for a simulated sample

The Density of Y

The marginal density of X_1 and X_2 is

$$\int_{-\infty}^{\infty} f(x_1^2 + x_2^2 + x_3^2)dx_3 = \int_y^{\infty} \frac{f(z)dz}{\sqrt{z-y}}.$$

So, the marginal density of Y is

$$g(y) = \pi \int_y^{\infty} \frac{f(z)dz}{\sqrt{z-y}}.$$

Estimating μ

The ψ's: Let

$$\psi(y) = \int v_3^2 f(y + x_3^2, v) d\boldsymbol{v} dx_3.$$

where $y = x_1^2 + x_2^2$ and $r^2 = y + x_3^2$. Thus,

$$E(V_3^2 | Y = y) = \frac{\pi\psi(y)}{g(y)},$$

and

$$\psi(y) = \int_y^\infty \frac{\mu(z)dz}{\sqrt{z-y}}.$$

Abel's Transformation

The Inverse: Let

$$\Psi(y) = \int_y^\infty \frac{\psi(t)dt}{\sqrt{t-y}}.$$

Then

$$\Psi(y) = \pi \int_y^\infty \mu(\sqrt{t})dt,$$

$$\mu(r) = -\frac{1}{\pi}\Psi'(r^2),$$

and

$$M(r) = \frac{2r^3}{G\pi f(r^2)}\Psi''(r^2).$$

Shape Restrictions: Ψ is convex and more.

The Naive Estimator

Given a sample $(X_{i,1}, X_{i,2}, V_{i,3})$, $i = 1, \cdots, n$, there is a *naive* estimator of Ψ:

$$\Psi^{\#}(t) = \frac{1}{n} \sum_{i:Y_i>t} \frac{V_{i,3}^2}{\sqrt{Y_i - t}}.$$

Properties

- Unbiased and consistent for each t
- Highly irregular as a function of t.

An Improved Estimator: Impose the shape restrictions.

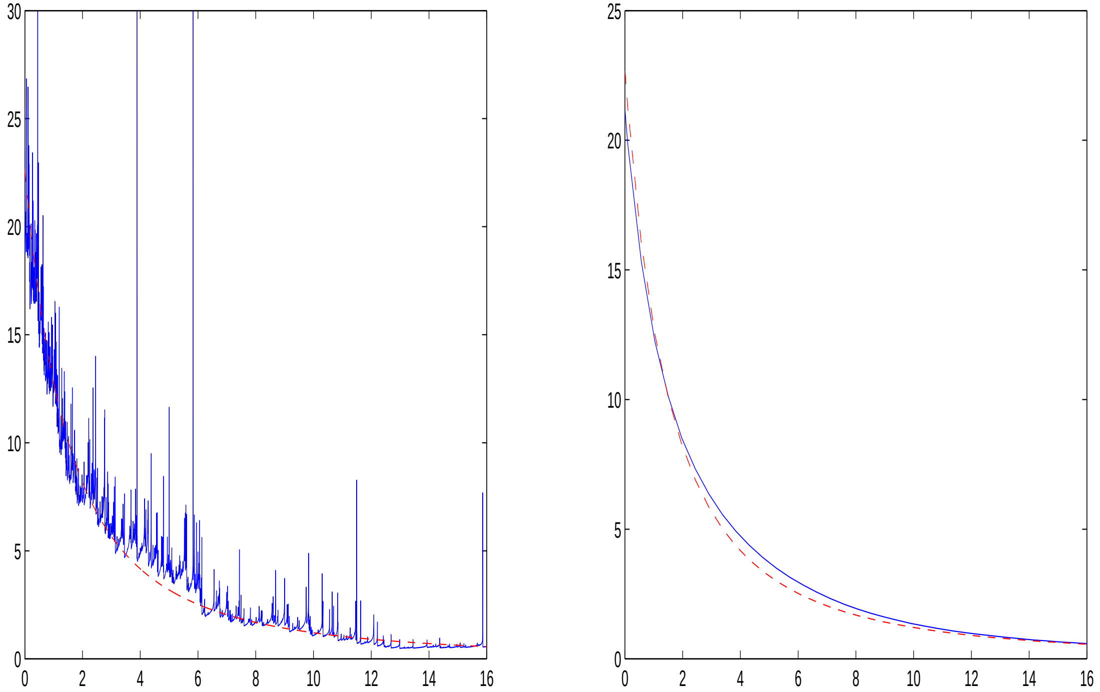

Figure 3: $\Psi^{\#}$ (left) and $\hat{\Psi}$ for a simulated sample.

Imposing the Shape Restrictions

Splines: Given $0 = r_0 < r_1 < \cdots < r_m$, suppose

$$M(r) = \theta_1 a(r) + \sum_{i=2}^{m} \theta_i (r - r_{i-1})_+^2$$

where $(x)_+^2 = \max[0, x]^2$, $a(r) \sim r^3$ as $r \to 0$, and $a(r) \sim r^2$ as $r \to \infty$.

Constraints: For $M(r)$ to be a bounded and increasing,

$$\sum_{i=1}^{m} \theta_i (r_j - r_{i-1})_+ \geq 0 \qquad (*)$$

for $j = 1, \cdots, m$ with equality when $i = m$; and in order for $\rho(r)$ to be decreasing, there additional linear constraints.

Continued: From Jeans' Equation,

$$\Psi(t) = \sum_{i=1}^{m} \theta_i \Gamma_i(t), \qquad (\dagger)$$

where

$$\Gamma_i(t) = \int_t^\infty \frac{G\pi f(s)}{2\sqrt{s^3}} (\sqrt{s} - r_{i-1})_+^2 ds, i \geq 2.$$

Minimize: the integral of $[\Psi(t) - \Psi^\#(t)]^2$ or, equivalently

$$\kappa = \int_0^\infty \Psi(t)^2 dt - 2 \int_0^\infty \Psi(t)\Psi^\#(t) dt,$$

among Ψ of the form ($\dagger$), subject to constraints on the θ_i.

Find:

$$\kappa = -2\sum_{i=1}^{m} w_i\theta_i + \sum_{i=1}^{m}\sum_{j=1}^{m} q_{ij}\theta_i\theta_j,$$

where

$$w_i = \int_0^\infty \Psi^{\#}(t)\Gamma_i(t)dt$$

and

$$q_{ij} = \int_0^\infty \Gamma_i(t)\Gamma_j(t)dt.$$

Quadratic Programming Problem: Get $\hat{\theta}$.

Estimation

The Improved Estimator: From $\hat{\theta}$, we get

$$\hat{\Psi}(t) = \sum_{i=1}^{m} \hat{\theta}_i \Gamma_i(t).$$

The Mass Estimator: From $\hat{\Psi}$, we get:

$$\hat{\mu}(r) = -\frac{1}{\pi}\hat{\Psi}'(r^2)$$

and

$$\hat{M}(r) = \frac{2r^3}{G\pi f(r^2)}\hat{\Psi}''(r^2)$$

Refs: Wang, Walker et. al. (2005, *A.J.L*), (2006, *A.J*), and *SCMA*.

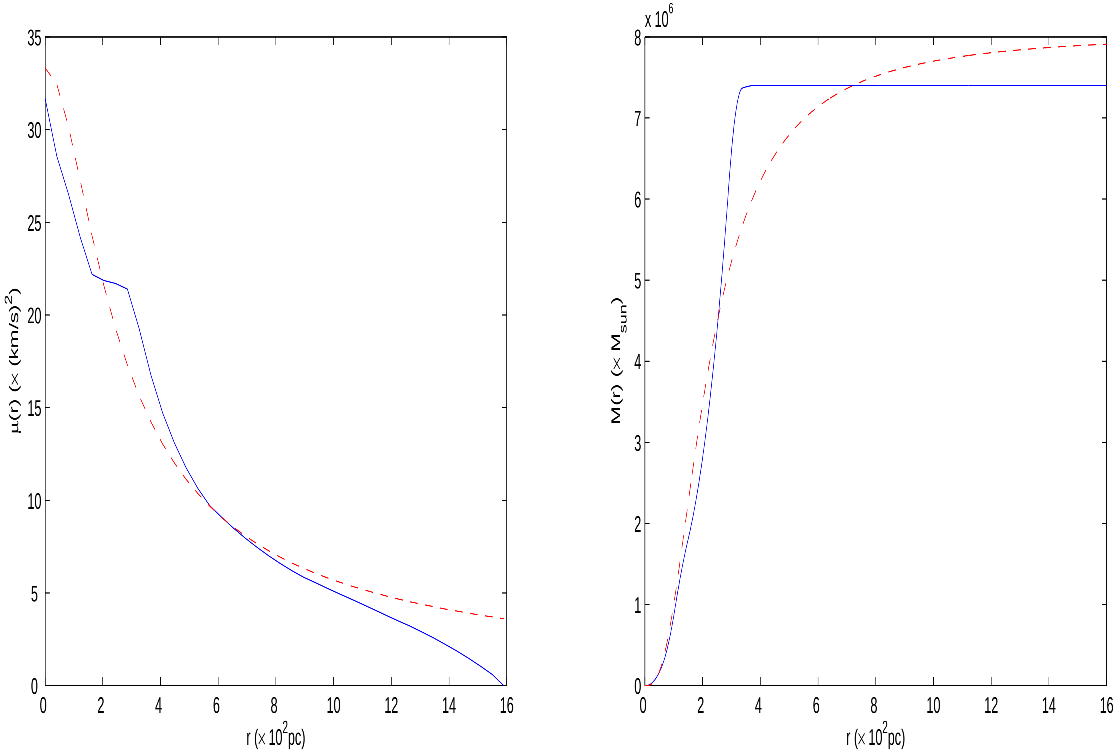

Figure 4: $\hat{\mu}$ (left) and $\hat{M}$ (right) for a simulated sample.

Complications

- Measurement Error: If $U_i = V_{i,3} + \sigma_i \epsilon_i$, replace $V_{i,3}^2$ by $U_i^2 - \sigma_i^2$ in $\Psi^{\#}$.

- Selection Effects: Let $w_0(x_1, x_2)$ the probability of observing a star at (x_1, x_2) (given that it is there), and

$$w(y) = \frac{1}{c} \int_{-\pi}^{\pi} w_0[\sqrt{y}\cos(\phi), \sqrt{y}\sin(\phi)]d\phi,$$

where c is a nomalizing constant. If w is known, replace

$$U_i^2 - \sigma_i^2 \qquad \text{by} \qquad \frac{U_i^2 - \sigma_i^2}{w(Y_i)}$$

in $\Psi^{\#}$. Otherwise, estimate w (possible).

Extensions

- More Data: 1000 Fornax Stars
- Sampling Distribution:
  - –Asymptotic Distributions
  - –Bootstrapping (Bodhisattva Sen)
- Cubic Splines
  - \- Get a convex programming problem.
  - \- Wang (In preparation)
- Anisotropy
  - \- Include one anisotropy parameter
  - \- Follow the same path.

Cubic Splines

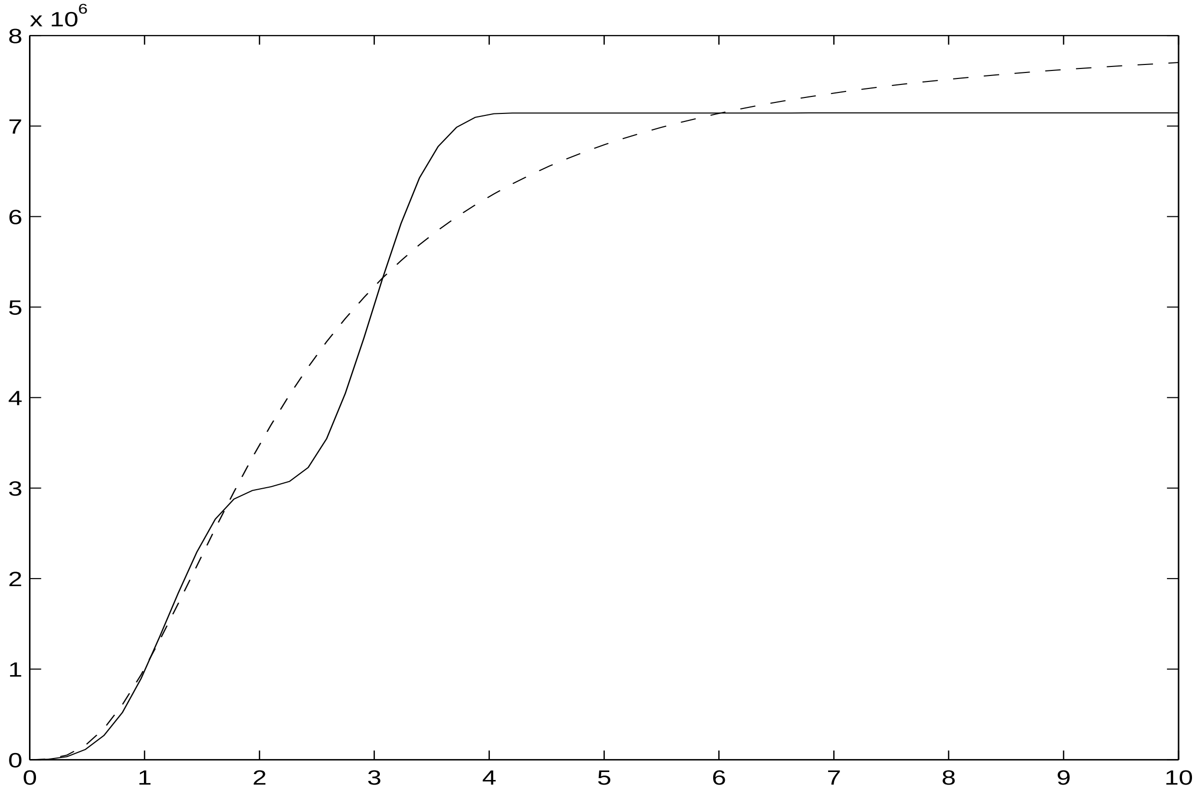

Figure 5: $\hat{M}$ with cubic splines.

Anisotropy

Jean's Equation: Assuming spatial symmetry and $\cdots$,

$$M(r) = -\frac{r^2}{Gf_0(r^2)}\left[\mu_r'(r) + 2\beta\frac{\mu_r(r)}{r}\right],$$

where

$$\mu_r(r) = \int (\frac{x \cdot \boldsymbol{v}}{r})^2 f(r^2, \boldsymbol{v}) d\boldsymbol{v}.$$

Major Complication: Line of sight velocity dispersions differ from the radial velocity dispersion, but the relationship can be determined.

Anisotropic Cases

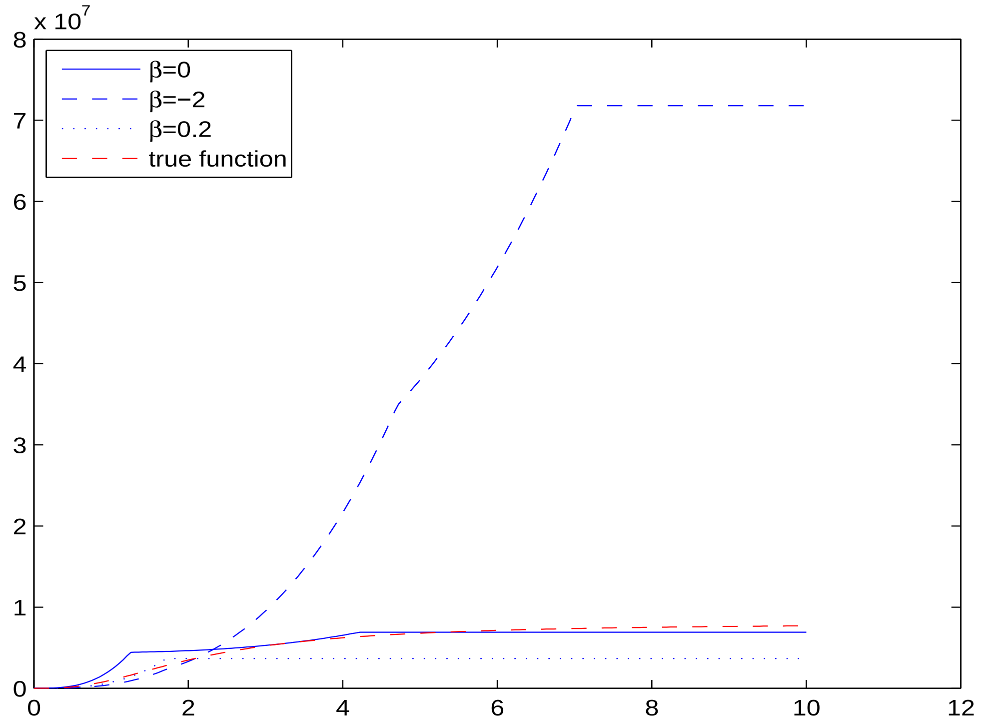

Figure 6: $\hat{M}$ with anisotropy.