Mathematical Pedagogy

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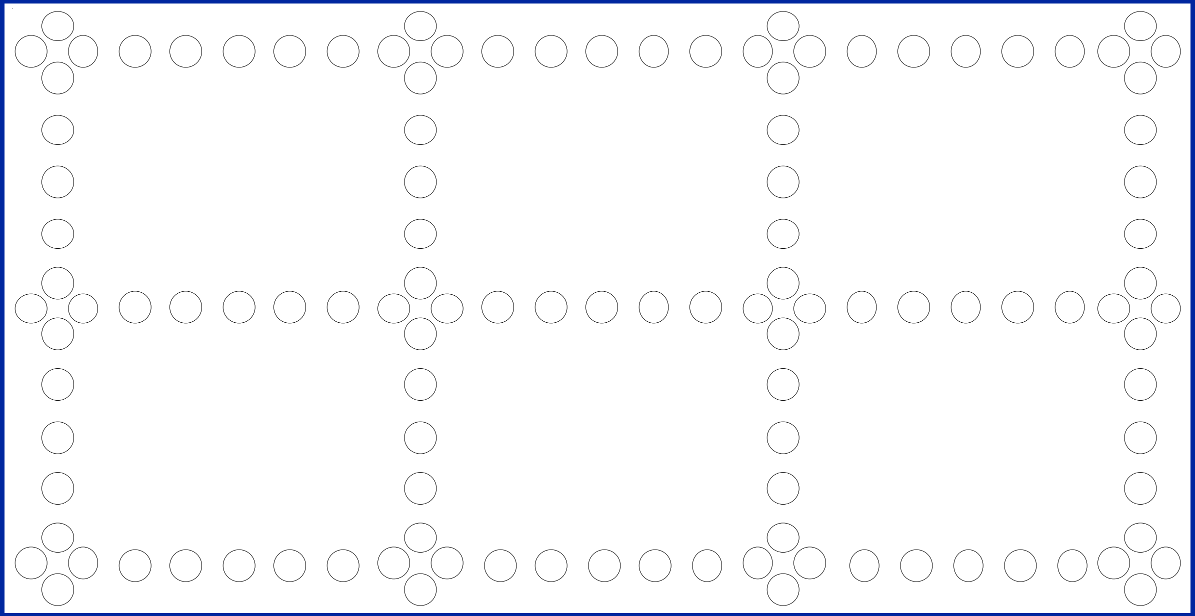
Definitions

- Mathematical Pedagogy
  - Strategies for teaching maths; useful constructs
- Mathematical Didactics
  - Tactics for teaching specific topics or concepts
- Pedagogical Mathematics
  - Mathematical explorations useful for, and arising from, pedagogical considerations

Perforations



How many holes for a sheet of *r* rows and *c* columns of stamps?

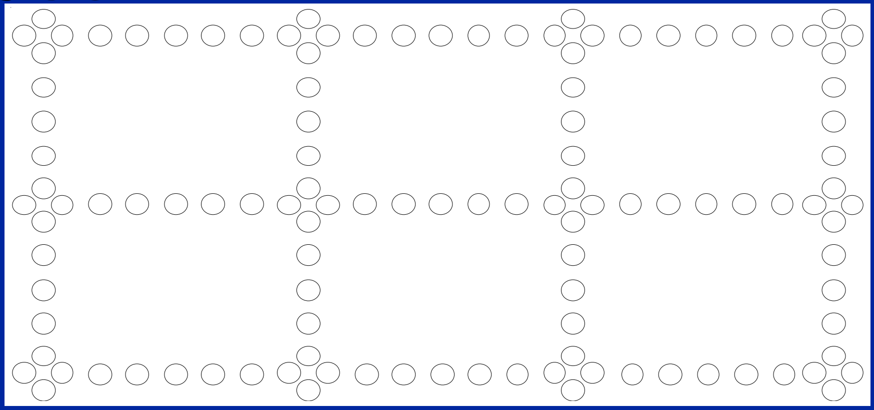
If someone claimed there were 228 perforations in a sheet, how could you check?

Possible Strategies

- Watch What You Do
  - Specialise but attend to what your body does as way of seeing & as source of generalisation
- Say What You See
  - Reveal/locate distinctions, relationships, properties, structure

Holding Wholes (gazing)
Discerning Details
Recognising Relationships
Perceiving Properties
Reasoning on the Basis of Properties

Perforations Generalised



- Dimensions of possible variation:

For Each Stamp:

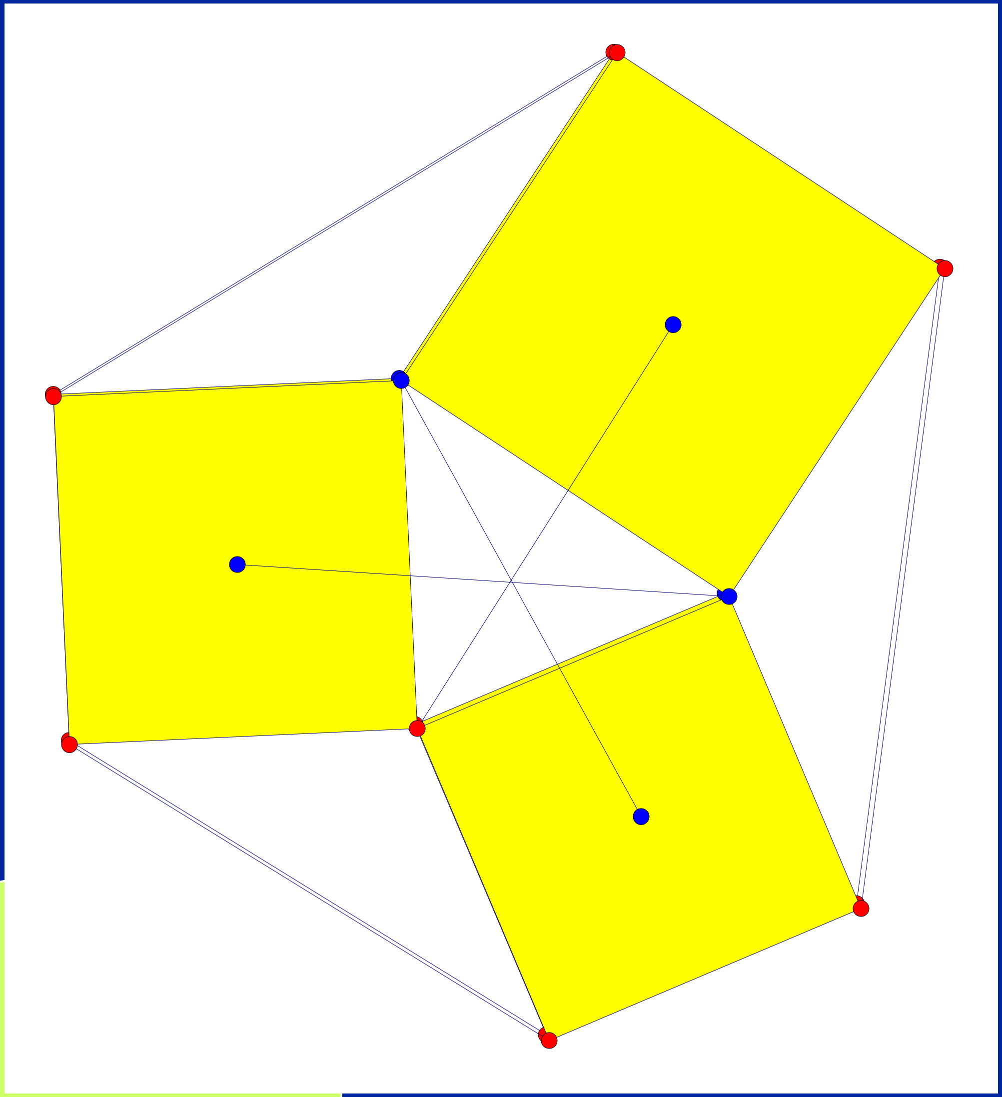
- number of horizontal perforations top & bottom
- number of vertical perforations, left & right
- number of perforations in the corners

Structural Generalisation: write down the number of perforations for R rows and C columns and characterise the numbers which can arise

Structured Variation Grids

- Pedagogical Offshoot

Vecten



Holding Wholes (gazing)
Discerning Details
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Chords

- ➨ Locus of midpoints of chords of a quartic?
- ➨ Locus of midpoints of fixed-width chords?

Chord Slopes

✓ family of chords with one end fixed: slope locus;
what happens as the length goes to zero?

✓ fixed width chords: slope locus;
what happens as the width goes to zero?

? chords of fixed length: slope locus;
what happens as the width goes to zero?

? envelope of slopes of chords through pt

Cubic Construction

- Construct a cubic for which the root-tangents are alternately perpendicular
- It seems a reasonable task, except that it is impossible!
- Why is it impossible?
  What sorts of constraints are acting?
- What about quartics?

Discovery

- Suppose a cubic has three distinct real roots.
- Then the sum of the cotangents of the root-angles is 0.
- More generally, for a polynomial of degree d, the sum of the products of the root-slopes taken d – 1 at a time is zero.

Extension

- Suppose a line cuts a polynomial of degree d > 1 in d distinct points.
- What is the sum of the cotangents of the angles the line makes with the polynomial at the intersection points?

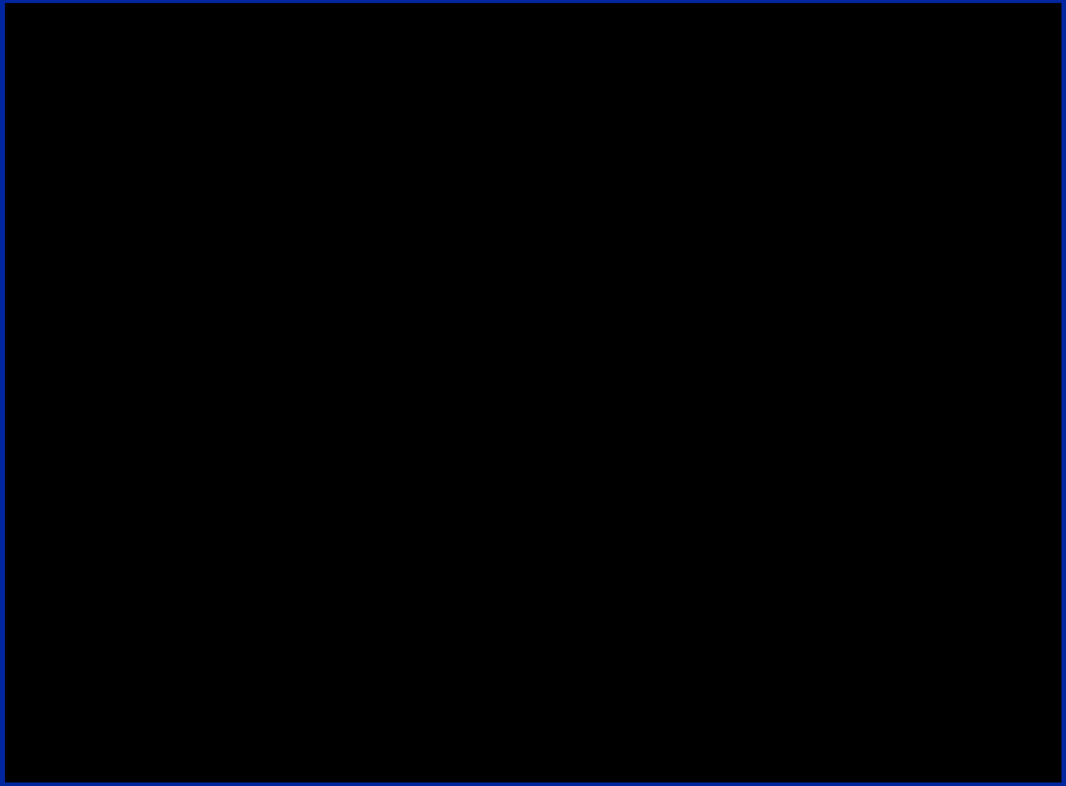
Cutting-Angles (1)

- **Let the line L(x) have slope m**

$$\cot(\alpha) = \frac{1}{\tan(\theta - \varphi)}$$

$$= \frac{1 + \tan(q)\tan(j)}{\tan(q) - \tan(j)}$$

$$= \frac{1 + p'(r_j)m}{p'(r_j) - m}$$



Cutting-Angles (2)

- Put f(x) = p(x)-L(x)
- We know that $\sum_{1}^{d} \frac{1}{f'(r_j)} = 0 = \sum_{1}^{\delta} \frac{1}{\pi'(\rho_\varphi) - \mu}$

- But the sum of the cots of the angles between line and function is

$$\sum_{1}^{\delta} \frac{1 + \mu \pi'(\rho_\varphi)}{\pi'(\rho_\varphi) - \mu} = \mu \sum_{1}^{\delta} 1 + \sum_{1}^{\delta} \frac{\mu^2 + 1}{\pi'(\rho_\varphi) - \mu} = \mu \delta$$

Root-slope polynomial

Given a polynomial $p(x) = \alpha \prod_{\kappa=1}^{\delta} (\xi - \rho_\kappa)$

Define the root-slope polynomial of p to be

$$p_\rho(\xi) = \alpha^\delta \prod_{\kappa=1}^{\delta} (\xi - \pi\ni(\rho_\kappa))$$

The constant term is $(-1)^d$ times the discriminant

which is the square of the product of the inter-rootal distances.

The coefficient of *x* is 0, which can be thought of as the sum of the reciprocals of the root-slopes, times the product of all the root-slopes.

Chordal Triangles

- Locus of centroids of chordal triangles?
- Locus of Circumcentre of chordal triangles with fixed chord widths?
- Locus of area of triangles with fixed chord widths?
- Limit of circumcentres?
- Limit of excentres of chordal triangle?
- Limit of centre of Bevan Circle?
- Limit of area/product of chord widths?

Mean Menger Curvature

- Given three points on a curve, the *Menger curvature* is the reciprocal of the radius of the circle through the three points
- Given three points on a function, they determine an interval on the x-axis. Is there a point in the interior of that interval, at which the curvature of the function is the Menger curvature of the three points?
- Try something easier first.

Rolle-Lagrange Mean-Parabola

- Given three points on a function but not on a straight line, there is a unique quadratic function through them.
- Is there a point in the interval spanned, to which some point on the parabola can be translated so as to match the function in value, slope and second derivative at that point?

Why 'Mean Value'?

Let f be integrable on [a, b].

The average (mean) value of f on [a, b] is $\dfrac{\int_a^b f}{\beta - \alpha}$

- **A sensible 'Mean Value' property (MVP):**

  There exists c in [a, b] for which $\int_a^b f = f(c)\,(b - a)$

- The usual 'Mean Value' Theorem(s) then become
  - Derivatives (being continuous) satisfy the MVP on every subinterval
  - Integrals of continuous functions satisfy the MVP on every subinterval

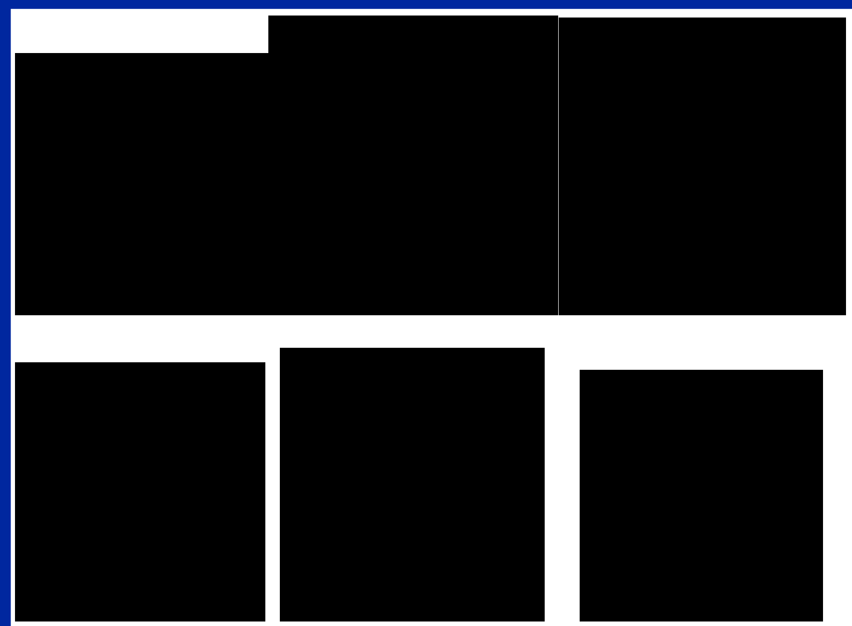
Mean Menger Curvature

Given a circle through three points (Menger circle), is there a point on the spanned interval with the same curvature?

Suppose f and g are both twice differentiable on $[a, b]$, that $k(f) < k(g)$ on $[a, b]$, and that $f(a) = g(a)$.

If $f(a)$ $g(a)$ then for all x in $(a, b]$, $f(x)$ $g(x)$: →

If $f(a)$ $g(a)$ then for at most one x in $(a, b]$, $f(x) = g(x)$. →



If g has constant curvature (is part of a circle) and $k(f) \neq k(g)$ on $[a, b]$ then f and g can intersect at most twice.

Therefore there must be a point s on $[a, b]$ at which $k(f(s))$ = Menger Curvature of the points a, b, c.

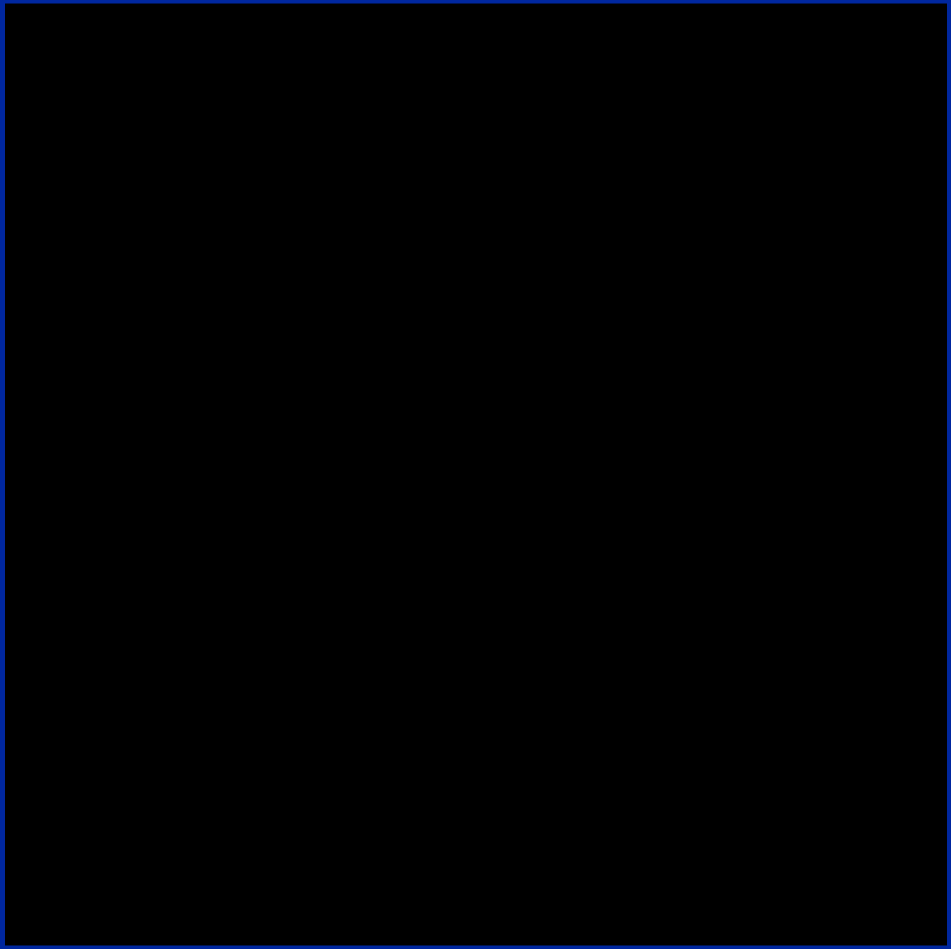
Cauchy Mean Value Theorem

Augustin Cauchy *1789-1857)*

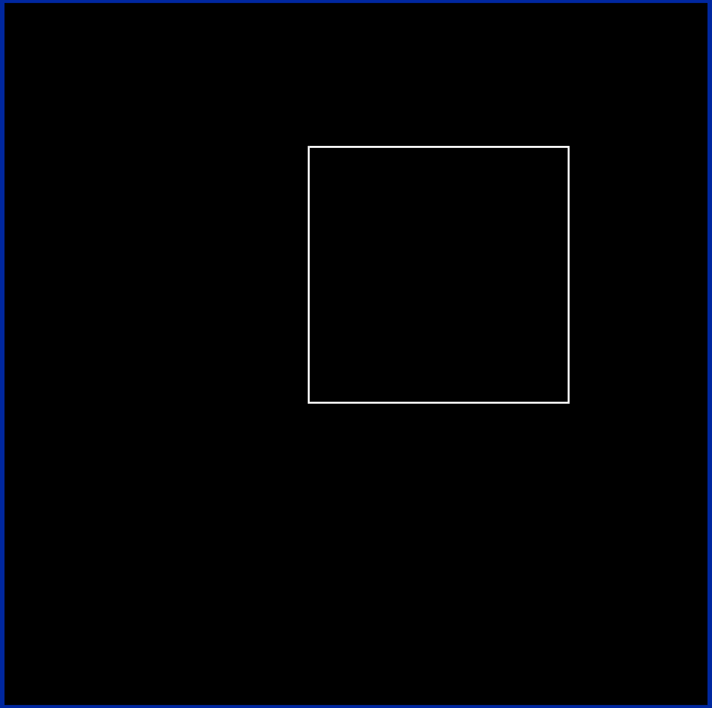
- **Let [*f(t), g(t)*] trace a differentiable curve in the plane**
  - **in each interval [*a, b*] there exists a point *s* at which**
    $[g(b) - g(a)]\, f'(s) = [f(b) - f(a)]\, g'(s)$

Cauchy Mean Menger Curvature?

Is there a Cauchy version of curvature
for curves in the plane?



NO!   Counter Example:



Procedural-Instrumental
Conceptual-Relational

- Human psyche is an interweaving of
  behaviour (enaction)
  emotion (affect)
  awareness (cognition)
- Behaviour is what is observable
- Teaching:
  - Expert awareness is transposed into instruction in behaviour
  - The more clearly the teacher indicates the behaviour expected, the easier it is for learners to display it without generating it from and for themselves

transposition didactique

didactic tension

Tasks & Teaching

- Tasks are only a vehicle for engaging in mathematical thinking
- Learners need to be guided, directed, prompted, and stimulated to make sense of their activity: to reflect
  - To manifest a reflection geometrically as a rotation, you need to move into a higher dimension!
  - The same applies to mathematical thinking!

Pedagogical Mathematics

- ‘Problems’ to explore probe learner awareness and comprehension, and afford opportunity to use their own powers to experience mathematical thinking
- These may not be at the cutting edge;
- Often they may be hidden in the undergrowth of mathematics of the past
- but they can be intriguing!
- Effective teaching of mathematics results in learners with a disposition to explore for themselves.

Further Reading

Mathematics Teaching Practice: a guide for university and college lecturers, Horwood Publishing. Mason J. (2002).

Mathematics as a Constructive Activity: learners generating examples. Erlbaum. Watson A. & Mason J. (2005).

Counter-Examples in Calculus. manuscript, Klymchuk, S. & Mason, J.

http://mcs.open.ac.uk/jhm3