Australian Government
Department of Defence
Defence Science and
Technology Organisation

Golay Complementary Sequences: A Tale of Two Heisenberg-Weyl Groups

Stephen Howard[1] Robert Calderbank[2] Bill Moran[3]

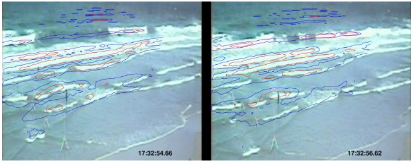
[1] Defence Science & Technology Organisation

[2] Princeton University

[3] The University of Melbourne

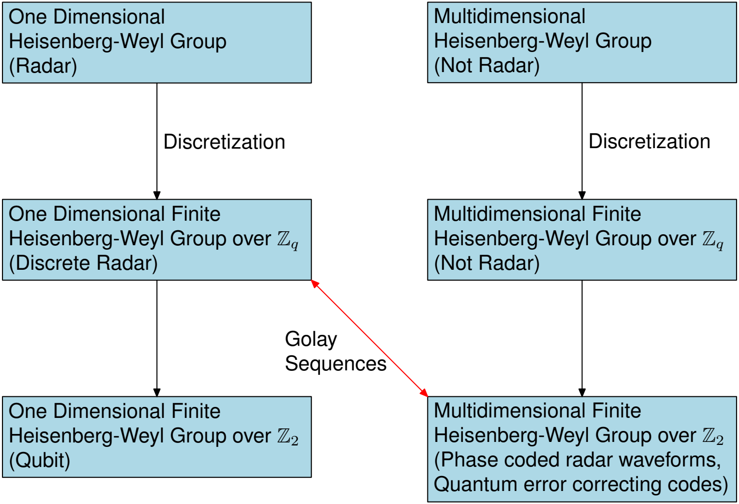
Supported by
Defense Advanced Research Projects Agency
Air Force Office of Scientific Research
Department of Education, Science & Training, International Science Linkages

Experimental Context



- Collaboration with the Naval Research Laboratory.
- Experimental platform where innovation in waveform design and signal processing can be tested in a relatively realistic littoral environment.
- Aim is to demonstrate environmental learning and to predict action on subsequent waveforms.

Heisenberg-Weyl Groups and Radar



One Dimensional Heisenberg-Weyl Group (Radar)

Multidimensional Heisenberg-Weyl Group (Not Radar)

Discretization

Discretization

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Discrete Radar)

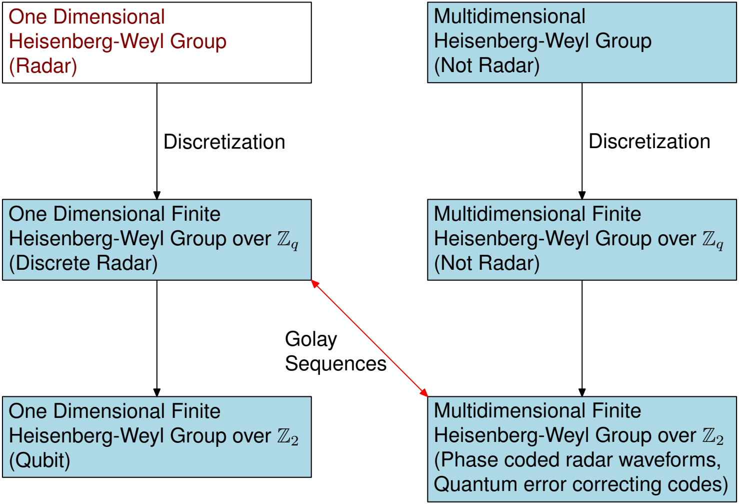
Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Not Radar)

Golay Sequences

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Qubit)

Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Phase coded radar waveforms, Quantum error correcting codes)

Heisenberg-Weyl Groups and Radar



One Dimensional
Heisenberg-Weyl Group
(Radar)

Multidimensional
Heisenberg-Weyl Group
(Not Radar)

Discretization

Discretization

One Dimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_q$
(Discrete Radar)

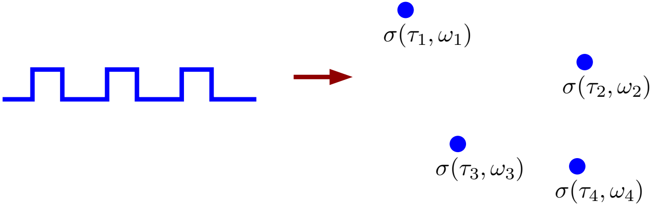
Multidimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_q$
(Not Radar)

Golay
Sequences

One Dimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_2$
(Qubit)

Multidimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_2$
(Phase coded radar waveforms,
Quantum error correcting codes)

Basic Narrow-bad Radar



Transmit a waveform $\phi \in \mathrm{L}^2(\mathbf{R})$.

This interacts with a point scatterer to give return

$$\psi(t) = \sigma(\tau,\omega)D(\tau,\omega)\phi\,(t) = \sigma(\tau,\omega)e^{i\omega t}\phi(t+\tau)$$

Radar Scene:

$$S = \sum_j \sigma(\tau_j,\omega_j)D(\tau_j,\omega_j)$$

Heisenberg-Weyl Operators:

$$D(\tau,\omega)\phi(t) = e^{-i\tau\omega}e^{i\omega t}\phi(t-\tau)$$

Basic Radar Signal Processing

- For optimal detection/estimation of a single point scatterer in white noise, the radar calculates, for return $\psi(t)$

$$\begin{aligned}(D(\tau,\omega)\phi,\psi) &= (D(\tau,\omega)\phi,S\phi)\\ &= \sum_j \sigma(\tau_j,\omega_j)\overline{\mathcal{A}(\tau-\tau_j,\omega-\omega_j)}e^{i\omega_j\tau}\end{aligned}$$

- Of fundamental importance is the *Ambiguity Function*

$$\mathcal{A}_\phi(\tau,\omega) = (\phi, D(\tau,\omega)\phi) = e^{-i\tau\omega}\int \overline{\phi(t)}\phi(t-\tau)e^{i\omega t}\,dt$$

  This is a function on the phase or range-Doppler space.

- *Moyal's Identity:*

$$\int |\mathcal{A}_\phi(\tau,\omega)|^2\,d\tau d\omega = ||\phi||^4$$

The Heisenberg-Weyl Group $\mathfrak{W}(R)$

- The continuous Heisenberg-Weyl group consists of the operators

$$\mathfrak{W}(R) = \{\lambda D(\tau,\omega) : |\lambda| = 1, (\tau,\omega) \in R^2\}$$

  This representaion is irreducible (Stone–Von Neumann)
- The group multiplication

$$D(\tau_1,\omega_1)D(\tau_2,\omega_2) = e^{i\tau_2\omega_1} D(\tau_1+\tau_2, \omega_1+\omega_2)$$

- The operators $D(\tau,\omega)$ do not commute:

$$D(\tau_1,\omega_1)D(\tau_2,\omega_2) = e^{i(\tau_2\omega_1-\tau_1\omega_2)} D(\tau_2,\omega_2)D(\tau_1,\omega_1)$$

- Phase space is $\mathfrak{W}(R)/Z(\mathfrak{W}(R)) = R^2$

Representation of Operators

- Every linear operator on $\mathrm{L}^2(\mathbf{R}^2)$ with $\mathrm{Tr}(S^\dagger S) < \infty$, i.e. *Hilbert-Schmidt class* $\mathfrak{H}$, can be represented as

$$S = \int_{R^2} s(\tau, \omega) D(\tau, \omega)\, d\tau d\omega$$

  with $s \in \mathrm{L}^2(\mathbf{R}^2)$.
- Isometry from $\mathfrak{H}$ to $\mathrm{L}^2(\mathbf{R}^2)$, given by

$$S \rightarrow s(\tau, \omega) = \mathrm{Tr}(D(\tau, \omega)^\dagger S)$$

  - *Weyl Transform*
- One dimensional projection Π_ϕ, i.e., $\Pi_\phi \psi = (\phi, \psi)\phi$ has representation

$$\Pi_\phi = \int_{R^2} \overline{\mathcal{A}_\phi(\tau, \omega)} D(\tau, \omega)\, d\tau d\omega$$

Commutative Subgroups and Waveforms

- Suppose we have a stationary background and we want to detect anything which moves:

$$S = S_B + S_T = \sum_j \sigma(\tau_j, 0) D(\tau_j, 0) + \sigma(\tau_0, \omega_0) D(\tau_0, \omega_0)$$

- The operators $H_F = \{D(\tau, 0) : \tau \in R\}$ commute, and so have common eigenvectors (functions) $\phi_\Omega(t) = e^{i\Omega t}$

$$D(\tau, 0)\phi_\Omega = e^{-i\Omega\tau}\phi_\Omega$$

  Then

$$S\phi_\Omega = \sum_j \sigma(\tau_j, 0) e^{-i\Omega\tau_j}\phi_\Omega + \sigma(\tau_0, \omega_0) e^{-i\tau_0\omega_0} e^{-i\tau_0\Omega}\phi_{\Omega+\omega_0}$$

- If we just want to measure range we could use “eigenvectors” of the commutative subgroup $H_D = \{D(0, \omega) : \omega \in R\}$, i.e., an impulse $\delta(t - t_0)$.

Linear Frequency Modulation – Chirps

- Another class of commutative subgroups take the form

$$H_\lambda = \{D(\tau, \lambda\tau) : \tau \in R\}$$

  for each $\lambda \in R$.
- The corresponding eigenvectors are linear chirps

$$\phi_{\Omega,\lambda}(t) = e^{i(\Omega t + \lambda t^2/2)}$$

- The autocorrelation function of a linear chirp is an impulse – *Pulse Compression*

Linear Frequency Modulation – Chirps

- Another class of commutative subgroups take the form

$$H_\lambda = \{D(\tau, \lambda\tau) : \tau \in R\}$$

Many useful waveforms are associated with (maximally) commutative subgroups of the Heisenberg-Weyl groups.

- The autocorrelation function of a linear chirp is an impulse – *Pulse Compression*

Symplectic Transforms

- These are the unitary transformations which have the effect

$$U(S)D(\tau,\omega)U(S)^\dagger = \mu D(s_{11}\tau + s_{12}\omega, s_{21}\tau + s_{22}\omega)$$

  with $|\mu| = 1$, such that the group structure

$$D(\tau_1,\omega_1)D(\tau_2,\omega_2) = e^{i(\tau_2\omega_1 - \tau_1\omega_2)} D(\tau_2,\omega_2)D(\tau_1,\omega_1)$$

  is preserved.
- This occurs if $\det S = 1$. This is $\mathsf{Sp}(2,R)$.
- If $S_\lambda = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ then

$$U(S_\lambda)D(\tau,\omega)U(S_\lambda)^\dagger = D(\tau, \omega + \lambda\tau)$$

  - This is LFM. For any waveform $U(S_\lambda)\phi(t) = e^{i\lambda t^2/2}\phi(t)$.
  - $U(S_\lambda)$ maps H_F to H_λ and eigenvectors of H_F to those of H_λ.

Symplectic Transforms

- These are the unitary transformations which have the effect

Many useful waveforms are associated with (maximally) commutative subgroups of the Heisenberg-Weyl groups.

Symplectic Transforms are how we get at them.

- If $S_\lambda = \begin{pmatrix} & \\ \lambda & 1 \end{pmatrix}$ then

$$U(S_\lambda)D(\tau,\omega)U(S_\lambda)^\dagger = D(\tau,\omega+\lambda\tau)$$

  - This is LFM. For any waveform $U(S_\lambda)\phi(t) = e^{i\lambda t^2/2}\phi(t)$.
  - $U(S_\lambda)$ maps H_F to H_λ and eigenvectors of H_F to those of H_λ.

Waveforms and Maximal Commutative Subgroups

$$\begin{tikzcd}
H_D \arrow[r, "F"] \arrow[d, dash] & H_F \arrow[d, dash] \\
\mathcal{F}_D \arrow[r, "F"'] & \mathcal{F}_F
\end{tikzcd}$$

$$\begin{tikzcd}
H_F \arrow[r, "U(S_\lambda)"] \arrow[d, dash] & H_\lambda \arrow[d, dash] \\
\mathcal{F}_F \arrow[r, "U(S_\lambda)"'] & \mathcal{F}_\lambda
\end{tikzcd}$$

Waveforms and Maximal Commutative Subgroups

$$\begin{tikzcd}
H_D = \{D(0,\omega) : \omega \in R\} \arrow[r, "F"] \arrow[d, dash] \arrow[dr, phantom, "(F\phi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_R e^{i\omega\tau}\phi(\tau)d\tau"] & H_F = \{D(\tau,0) : \tau \in R\} \arrow[d, dash] \\
\{\boldsymbol{e}_\tau = \delta_\tau : \tau \in R\} \arrow[r, "F"'] & \{\boldsymbol{e}_{\hat\omega}(t) = \frac{1}{\sqrt{2\pi}} e^{i\omega t}\}
\end{tikzcd}$$

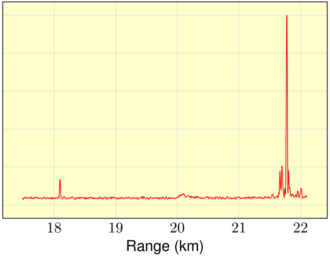
$$\begin{tikzcd}
H_F = \{D(\tau,0) : \tau \in R\} \arrow[r, "U(S_\lambda)"] \arrow[d, dash] \arrow[dr, phantom, "(U(S_\lambda)\phi)(t) = e^{i\lambda\tau^2/2}\phi(t)"] & H_\lambda = \{D(\tau,\lambda\tau) : \tau \in R\} \arrow[d, dash] \\
\{\boldsymbol{e}_{\hat\omega}(t) = \frac{1}{\sqrt{2\pi}} e^{i\omega t}\} \arrow[r, "U(S_\lambda)"'] & \{\boldsymbol{e}_{\lambda,\hat\omega}(t) = \frac{1}{\sqrt{2\pi}} e^{i\lambda t^2/2} e^{i\omega t} : \omega \in R\}
\end{tikzcd}$$

You Want to Transmit What?

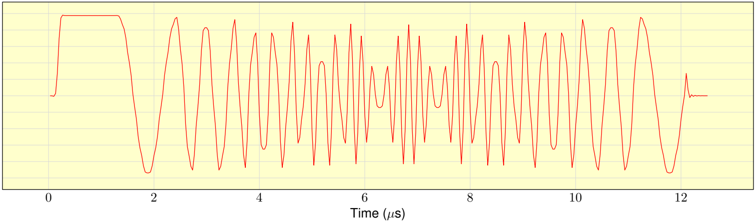
- Of course, our waveforms must have finite duration and peak power.
- But there are other constraints:
  - If the transmitter and receiver are not isolated the waveform must be pulsed.
  - The waveform will make the most efficient use of radar transmit resources if they have peak to average power ratios close to 1.
- Unimodular waveforms do this best.
  - Linear and non-linear frequency modulation.
  - Unimodular complex sequences phase modulating a carrier wave.

The Island, the Ship and the Side Lobe

- If we transmit a pulse (here Frank coded) the autocorrelation has side lobes.
- The side lobe from a large object can interfere with the detection of a small target – *Clutter*

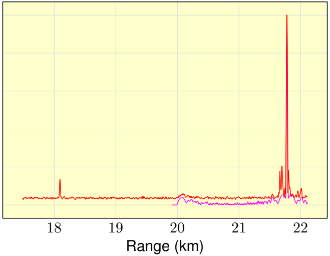


Frank coded waveform

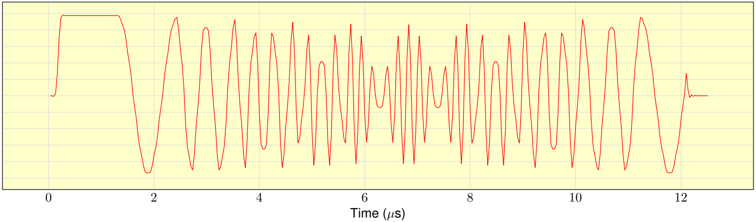


The Island, the Ship and the Side Lobe

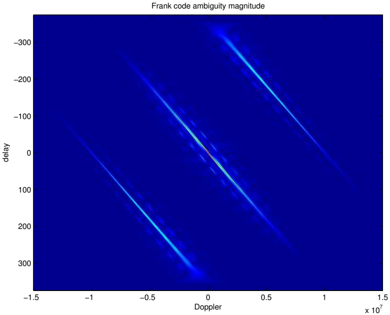
- If we transmit a pulse (here Frank coded) the autocorrelation has side lobes.
- The side lobe from a large object can interfere with the detection of a small target – *Clutter*

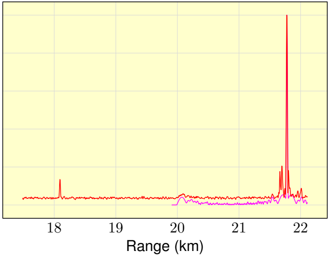


Frank coded waveform

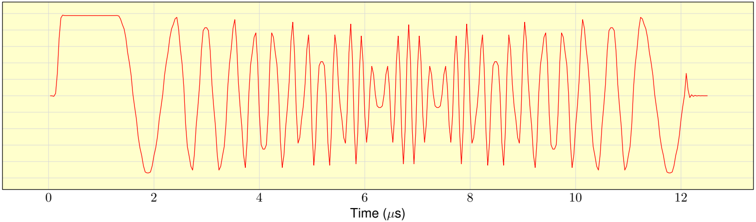


The Island, the Ship and the Side Lobe





Frank coded waveform



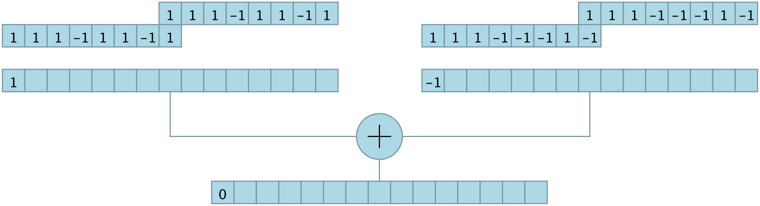
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



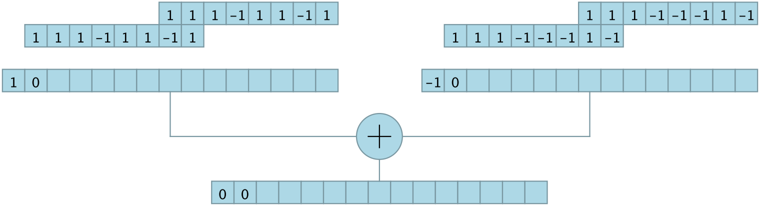
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathsf{corr}_k(\mathbf{x}) + \mathsf{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



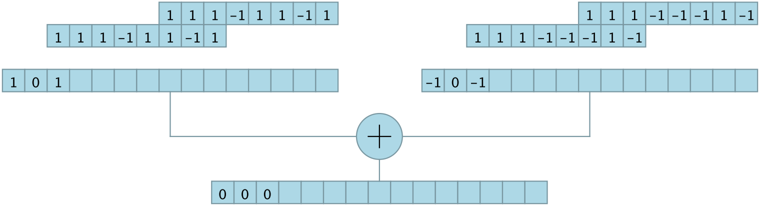
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1
1 1 1 -1 1 1 -1 1
1 0 1

1 1 1 -1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1
-1 0 -1

0 0 0

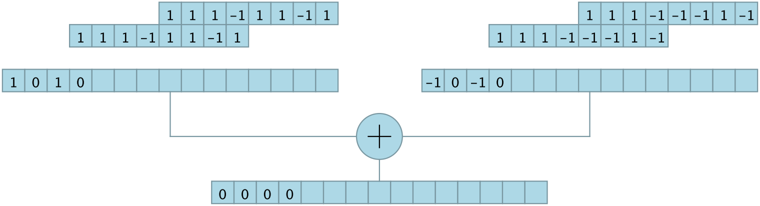
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathsf{corr}_k(\mathbf{x}) + \mathsf{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



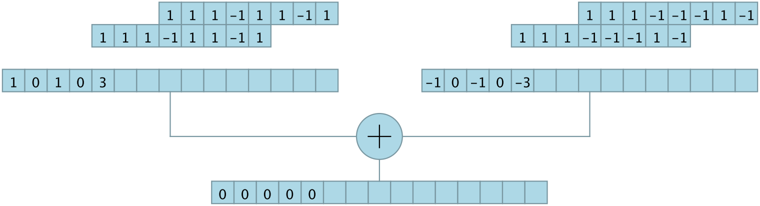
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathsf{corr}_k(\mathbf{x}) + \mathsf{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 −1 1 1 −1 1
1 1 1 −1 1 1 −1 1
1 0 1 0 3

1 1 1 −1 −1 −1 1 −1
1 1 1 −1 −1 −1 1 −1
−1 0 −1 0 −3

$+$

0 0 0 0 0

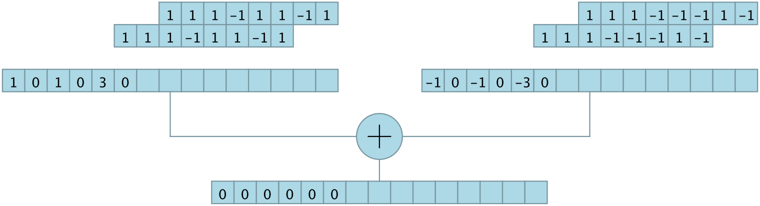
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



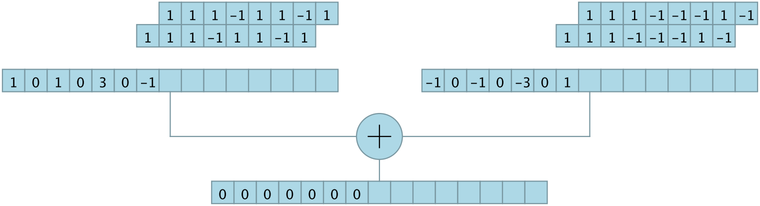
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1
1 1 1 -1 1 1 -1 1
1 0 1 0 3 0 -1

1 1 1 -1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1
-1 0 -1 0 -3 0 1

$+$

0 0 0 0 0 0 0

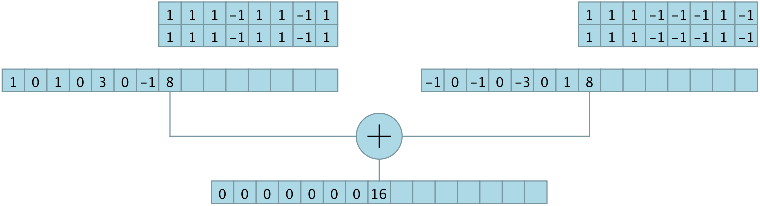
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\text{corr}_k(\mathbf{x}) + \text{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1	1	1	-1	1	1	-1	1
1	1	1	-1	1	1	-1	1

1	0	1	0	3	0	-1	8							

1	1	1	-1	-1	-1	1	-1
1	1	1	-1	-1	-1	1	-1

-1	0	-1	0	-3	0	1	8							

0	0	0	0	0	0	0	16							

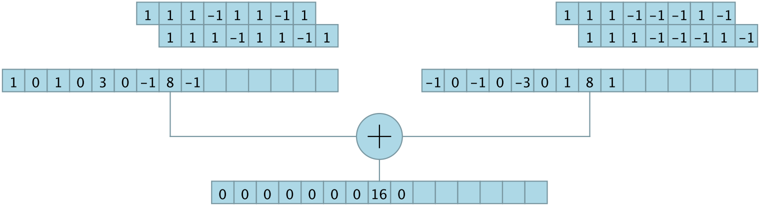
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 −1 1 1 −1 1
1 1 1 −1 1 1 −1 1

1 0 1 0 3 0 -1 8 -1

1 1 1 −1 −1 −1 1 −1
1 1 1 −1 −1 −1 1 −1

−1 0 −1 0 -3 0 1 8 1

+

0 0 0 0 0 0 0 16 0

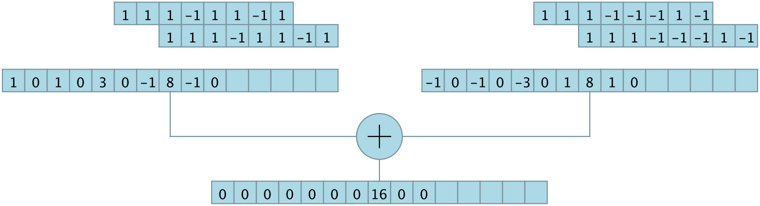
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathsf{corr}_k(\mathbf{x}) + \mathsf{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1
1 1 1 -1 1 1 -1 1

1 0 1 0 3 0 -1 8 -1 0

1 1 1 -1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1

-1 0 -1 0 -3 0 1 8 1 0

+

0 0 0 0 0 0 0 16 0 0

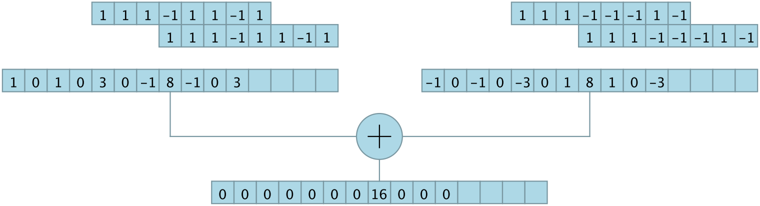
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1
1 1 1 -1 1 1 -1 1

1 0 1 0 3 0 -1 8 -1 0 3

1 1 1 -1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1

-1 0 -1 0 -3 0 1 8 1 0 -3

0 0 0 0 0 0 0 16 0 0 0

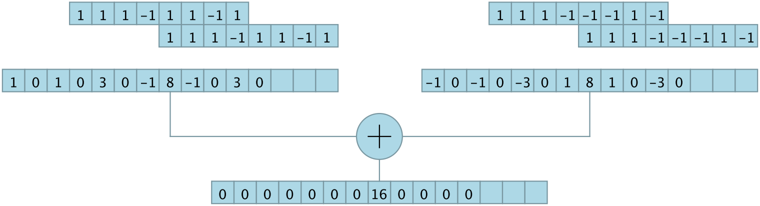
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1

1 1 1 -1 1 1 -1 1

1 0 1 0 3 0 -1 8 -1 0 3 0

1 1 1 -1 -1 -1 1 -1

1 1 1 -1 -1 -1 1 -1

-1 0 -1 0 -3 0 1 8 1 0 -3 0

0 0 0 0 0 0 0 16 0 0 0 0

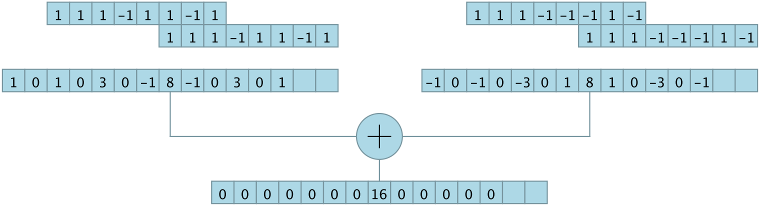
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathsf{corr}_k(\mathbf{x}) + \mathsf{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1
1 1 1 -1 1 1 -1 1

1 0 1 0 3 0 -1 8 -1 0 3 0 1

1 1 1 -1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1

-1 0 -1 0 -3 0 1 8 1 0 -3 0 -1

0 0 0 0 0 0 0 16 0 0 0 0 0

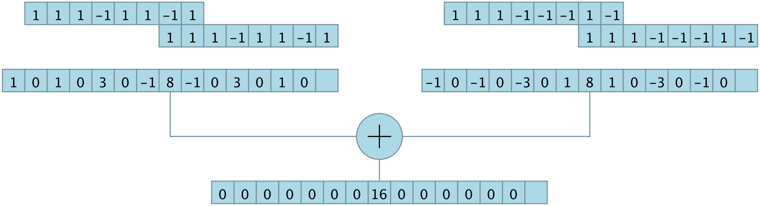
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathsf{corr}_k(\mathbf{x}) + \mathsf{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 −1 1 1 −1 1
1 1 1 −1 1 1 −1 1

1 0 1 0 3 0 −1 8 −1 0 3 0 1 0

1 1 1 −1 −1 −1 1 −1
1 1 1 −1 −1 −1 1 −1

−1 0 −1 0 −3 0 1 8 1 0 −3 0 −1 0

$+$

0 0 0 0 0 0 0 16 0 0 0 0 0 0

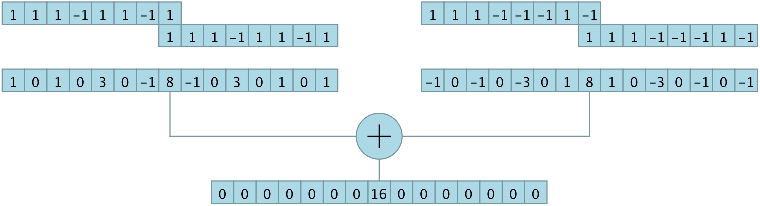
Golay Complementary Sequences

Definition

Two length N unimodular sequences of complex numbers $\mathbf{x}$ and $\mathbf{y}$ are Golay complementary if the sum of their auto-correlation functions satisfies,

$$\mathrm{corr}_k(\mathbf{x}) + \mathrm{corr}_k(\mathbf{y}) = 2N\delta_{k,0}$$

for $k = -(N-1), \cdots, (N-1)$.



1 1 1 -1 1 1 -1 1
1 1 1 -1 1 1 -1 1

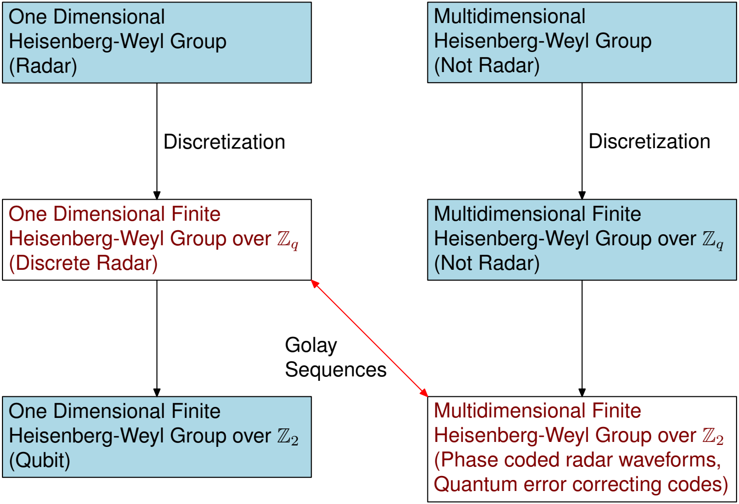
1 0 1 0 3 0 -1 8 -1 0 3 0 1 0 1

1 1 1 -1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1

-1 0 -1 0 -3 0 1 8 1 0 -3 0 -1 0 -1

0 0 0 0 0 0 0 16 0 0 0 0 0 0 0

Finite Heisenberg-Weyl Groups



One Dimensional Heisenberg-Weyl Group (Radar)

Multidimensional Heisenberg-Weyl Group (Not Radar)

Discretization

Discretization

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Discrete Radar)

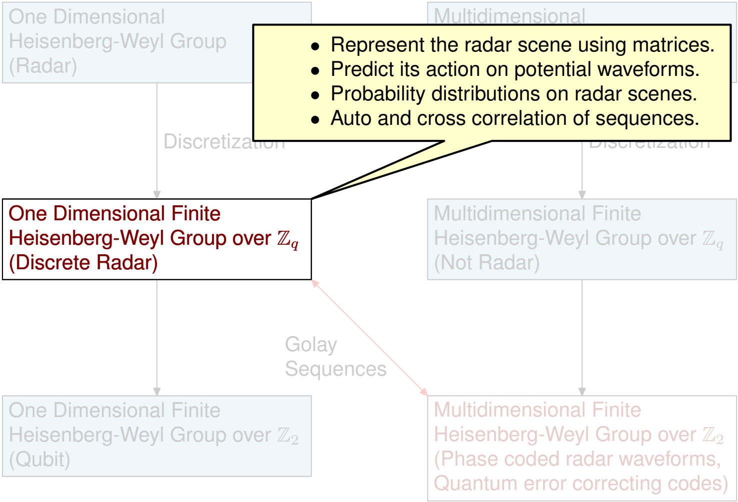
Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Not Radar)

Golay Sequences

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Qubit)

Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Phase coded radar waveforms, Quantum error correcting codes)

Finite Heisenberg-Weyl Groups



- Represent the radar scene using matrices.
- Predict its action on potential waveforms.
- Probability distributions on radar scenes.
- Auto and cross correlation of sequences.

One Dimensional Heisenberg-Weyl Group (Radar)

Multidimensional

Discretization

Discretization

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Discrete Radar)

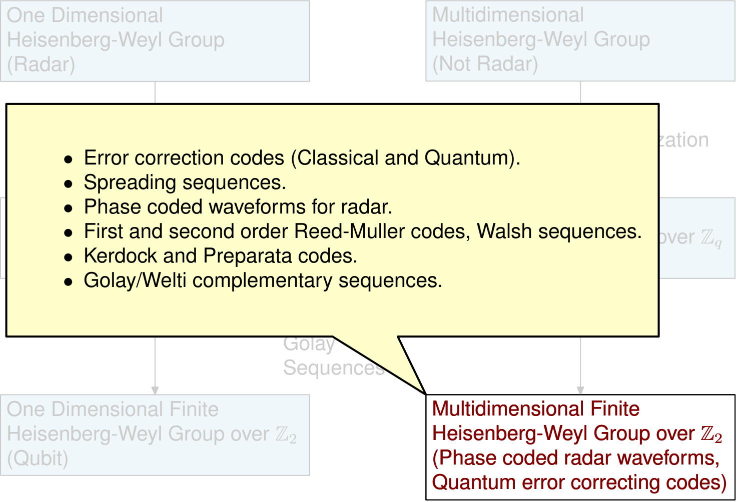
Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Not Radar)

Golay Sequences

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Qubit)

Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Phase coded radar waveforms, Quantum error correcting codes)

Finite Heisenberg-Weyl Groups



One Dimensional Heisenberg-Weyl Group (Radar)

Multidimensional Heisenberg-Weyl Group (Not Radar)

- Error correction codes (Classical and Quantum).
- Spreading sequences.
- Phase coded waveforms for radar.
- First and second order Reed-Muller codes, Walsh sequences.
- Kerdock and Preparata codes.
- Golay/Welti complementary sequences.

zation

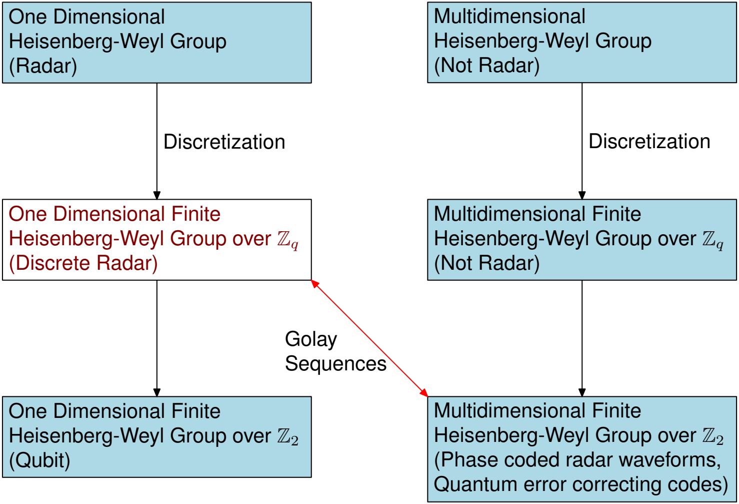
over $\mathbb{Z}_q$

Golay Sequences

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Qubit)

Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Phase coded radar waveforms, Quantum error correcting codes)

Heisenberg-Weyl Groups and Radar



One Dimensional
Heisenberg-Weyl Group
(Radar)

Discretization

One Dimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_q$
(Discrete Radar)

One Dimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_2$
(Qubit)

Multidimensional
Heisenberg-Weyl Group
(Not Radar)

Discretization

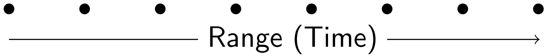
Multidimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_q$
(Not Radar)

Golay
Sequences

Multidimensional Finite
Heisenberg-Weyl Group over $\mathbb{Z}_2$
(Phase coded radar waveforms,
Quantum error correcting codes)

Discrete Finite Radar

For discrete finite radar the configuration space $\mathbb{Z}_N$ consists of a large number N of discrete times.



Waveforms: Square summable complex valued functions ϕ on $\mathbb{Z}_N$, $\mathfrak{H} = \mathrm{L}^2(\mathbb{Z}_N)$.

Point Scatterers: The action of a point scatterer with unit crossection is described by the operator $D(\tau, \nu)$

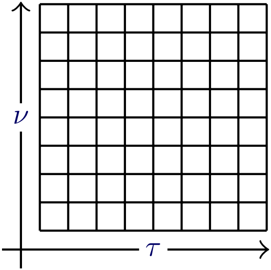
$$(D(\tau, \nu)\phi)_t = \omega^{\nu t}\phi_{t-\tau}$$

where $\omega = \exp(2\pi i/N)$.

Operator View of Discrete Radar

Radar scene operator S

$$S = \sum_{\tau,\nu\in\mathbb{Z}_N} \sigma(\tau,\nu)D(\tau,\nu)$$



For optimal detection/estimation of a single point scatterer in white noise, the radar calculates, for return ψ

$$\begin{aligned}(D(\tau,\nu)\phi,\psi) &= (D(\tau,\nu)\phi, S\phi)\\ &= \sum_j \sigma(\tau_j,\omega_j)\overline{A(\tau-\tau_j,\nu-\nu_j)}\omega^{\nu_j\tau}\end{aligned}$$

Ambiguity Function for 1-D Finite Heisenberg-Weyl Groups

- The ambiguity function of a waveform ϕ

$$\mathcal{A}_\phi(\tau,\nu) = (\phi, D(\tau,\nu)\phi) = \sum_{t\in\mathbb{Z}_N} \overline{\phi(n)}\phi(t-\tau)\omega^{\nu t}$$

- Moyal's identity:

$$\frac{1}{N}\sum_{\tau,\nu\in\mathbb{Z}_N} |\mathcal{A}_\phi(\tau,\nu)|^2 = \|\,\phi\,\|^4$$

The Heisenberg-Weyl Group $\mathfrak{W}(\mathbb{Z}_N)$

- The 1-D Finite Heisenberg-Weyl group consists of the operators

$$\mathfrak{W}(\mathbb{Z}_N) = \{\lambda D(\tau,\nu) : |\lambda| = 1, (\tau,\nu) \in \mathbb{Z}_N \times \mathbb{Z}_N\}$$

- This is a unitary irreducible representation on $\mathfrak{H}$
- The group multiplication is

$$D(\tau_1,\nu_1)D(\tau_2,\nu_2) = \omega^{\tau_2\nu_1} D(\tau_1+\tau_2,\nu_1+\nu_2)$$

- The operators $D(\tau,\nu)$ do not commute:

$$D(\tau_1,\nu_1)D(\tau_2,\nu_2) = \omega^{(\tau_2\nu_1-\tau_1\nu_2)} D(\tau_2,\nu_2)D(\tau_1,\nu_1)$$

Representation of Operators

- Every linear operator on $\mathrm{L}^2(\mathbb{Z}_N)$ can be represented a

$$S = \sum_{\tau,\nu \in \mathbb{Z}_N} s(\tau,\nu) D(\tau,\nu)$$

  with $s \in \mathrm{L}^2(\mathbb{Z}_N \times \mathbb{Z}_N)$.
- Isometry from $\mathfrak{O}$ to $\mathrm{L}^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, given by

$$S \to s(\tau,\nu) = \mathrm{Tr}(D(\tau,\nu)^\dagger S)$$

  - *Weyl Transform*
- One dimensional projection Π_ϕ, i.e., $\Pi_\phi \psi = (\phi,\psi)\phi$ has representation

$$\Pi_\phi = \sum_{\tau,\nu \in \mathbb{Z}_N} \overline{\mathcal{A}_\phi(\tau,\nu)} D(\tau,\nu)$$

Waveforms for Discrete Radar

$$\begin{tikzcd}
H_D \arrow[r, "F"] \arrow[d, dash] & H_F \arrow[d, dash] \\
\mathcal{F}_D \arrow[r, "F"'] & \mathcal{F}_F
\end{tikzcd}$$

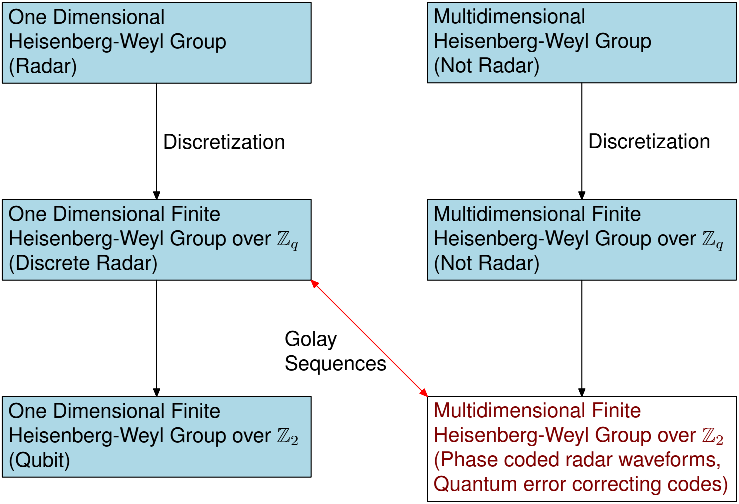
$$\begin{tikzcd}
H_F \arrow[r, "U(\ell)"] \arrow[d, dash] & H_\ell \arrow[d, dash] \\
\mathcal{F}_F \arrow[r, "U(\ell)"'] & \mathcal{F}_\ell
\end{tikzcd}$$

Waveforms for Discrete Radar

$$\begin{tikzcd}
\{D(0,\nu) : \nu \in \mathbb{Z}_n\} \arrow[rr, "F"] \arrow[dd, dash] & & \{D(\tau,0) : \tau \in \mathbb{Z}_n\} \arrow[dd, dash] \\
& (F\phi)_\nu = \frac{1}{\sqrt{n}} \sum_{\tau \in \mathbb{Z}_n} \omega^{\nu\tau} \phi_\tau & \\
\{\boldsymbol{e}_\tau : \tau \in \mathbb{Z}_n\} \arrow[rr, "F"'] & & \{\boldsymbol{e}_{\hat{\nu}} = \frac{1}{\sqrt{n}} \sum_{\tau \in \mathbb{Z}_n} \omega^{\nu\tau} \boldsymbol{e}_\tau : \nu \in \mathbb{Z}_n\}
\end{tikzcd}$$

$$\begin{tikzcd}
H_F = \{D(\tau,0) : \tau \in \mathbb{Z}_n\} \arrow[rr, "U(\ell)"] \arrow[dd, dash] & & H_\ell = \{D(\tau,2\ell\tau) : \tau \in \mathbb{Z}_n\} \arrow[dd, dash] \\
& (U(\ell)\phi)_\tau = \omega^{\ell\tau^2} \phi_\tau & \\
\{\boldsymbol{e}_{\hat{\nu}} = \frac{1}{\sqrt{n}} \sum_{\tau \in \mathbb{Z}_n} \omega^{\nu\tau} \boldsymbol{e}_\tau : \nu \in \mathbb{Z}_n\} \arrow[rr, "U(\ell)"'] & & \{\boldsymbol{e}_{\ell,\hat{\nu}} = \frac{1}{\sqrt{n}} \sum_{\tau \in \mathbb{Z}_n} \omega^{\ell\tau^2} \omega^{\nu\tau} \boldsymbol{e}_\tau : \nu \in \mathbb{Z}_n\}
\end{tikzcd}$$

Heisenberg-Weyl Groups and Radar



One Dimensional Heisenberg-Weyl Group (Radar)

Multidimensional Heisenberg-Weyl Group (Not Radar)

Discretization

Discretization

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Discrete Radar)

Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_q$ (Not Radar)

Golay Sequences

One Dimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Qubit)

Multidimensional Finite Heisenberg-Weyl Group over $\mathbb{Z}_2$ (Phase coded radar waveforms, Quantum error correcting codes)

The Heisenberg-Weyl Group Associated with Quantum Error Correction

Hilbert Space: Dirac basis $e_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}_2^m$, labeled by $N = 2^m$ states in an m qubit quantum system

Unitary Operators:

$$D(\mathbf{a}, \mathbf{0}) : e_{\mathbf{c}} \to e_{\mathbf{c} \oplus \mathbf{a}} \quad \text{bit errors in } m \text{ qubits}$$

$$D(\mathbf{0}, \mathbf{b}) : e_{\mathbf{c}} \to (-1)^{\mathbf{b} \cdot \mathbf{c}} e_{\mathbf{c}} \quad \text{phase errors in } m \text{ qubits}$$

where $\oplus$ denotes addition modulo 2.

For a vector (sequence) ϕ we have

$$D(\mathbf{a}, \mathbf{b})\phi(\mathbf{c}) = (-1)^{\mathbf{b} \cdot \mathbf{c}} \phi(\mathbf{c} \oplus \mathbf{a})$$

Finite Heisenberg-Weyl Group $\mathfrak{W}(\mathbb{Z}_2^m)$

- The finite Heisenberg-Weyl group over $\mathbb{Z}_2^m$ consists of the operators

$$\mathfrak{W}(\mathbb{Z}_2^m) = \{i^\lambda D(\mathbf{a},\mathbf{b}) : |\lambda \in \mathbb{Z}_4, (\mathbf{a},\mathbf{b}) \in \mathbb{Z}_2^m \times \mathbb{Z}_2^m\}$$

- The group multiplication

$$D(\mathbf{a},\mathbf{b})D(\mathbf{a}',\mathbf{b}') = (-1)^{\mathbf{b}\cdot\mathbf{a}'} D(\mathbf{a}\oplus\mathbf{a}',\mathbf{b}\oplus\mathbf{b}')$$

- Operators $D(\mathbf{a},\mathbf{b})$ either commute or anticommute

$$D(\mathbf{a},\mathbf{b})D(\mathbf{a}',\mathbf{b}') = (-1)^{\mathbf{b}\cdot\mathbf{a}'+\mathbf{b}'\cdot\mathbf{a}} D(\mathbf{a}',\mathbf{b}')D(\mathbf{a},\mathbf{b})$$

- $D(\mathbf{a},\mathbf{b})$ and $D(\mathbf{a}',\mathbf{b}')$ commute iff

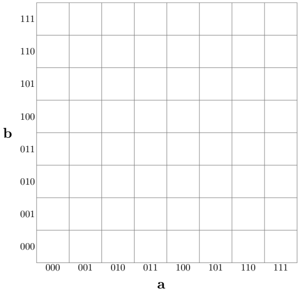
$$\big\langle(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')\big\rangle \equiv \mathbf{b}\cdot\mathbf{a}' + \mathbf{b}'\cdot\mathbf{a} = 0$$

Phase Space $W(\mathbb{Z}_2^m)$

- The phase space associated with $\mathfrak{W}(\mathbb{Z}_2^m)$ is

$$W(\mathbb{Z}_2^m) = \mathfrak{W}(\mathbb{Z}_2^m)/Z(\mathfrak{W}(\mathbb{Z}_2^m))$$

This is just $\mathbb{Z}_2^m \times \mathbb{Z}_2^m$



- Each commutative subgroup $H = \{D(\mathbf{a},\mathbf{b}) | (\mathbf{a},\mathbf{b}) \in \overline{H}\}$ corresponds to an isotropic subspace $\overline{H}$, i.e., for all $(\mathbf{a},\mathbf{b}), (\mathbf{a}',\mathbf{b}') \in \overline{H}$

$$\langle(\mathbf{a},\mathbf{b}), (\mathbf{a}',\mathbf{b}')\rangle = 0$$

Waveforms/Sequences

$$\begin{tikzcd}
H_D = \{D(0,\mathbf{b}) : \mathbf{b} \in \mathbb{Z}_2^m\} \arrow[r, "F"] \arrow[d, dash] & H_F = \{D(\mathbf{a},0) : \mathbf{a} \in \mathbb{Z}_2^m\} \arrow[d, dash] \\
\{e_{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}_2^m\} \arrow[r, "F"'] & \{e_{\hat{\mathbf{b}}} = \frac{1}{\sqrt{2^m}} \sum_{\mathbf{a}\in\mathbb{Z}_2^m} (-1)^{\mathbf{b}\cdot\mathbf{a}} e_{\mathbf{a}} : \mathbf{b} \in \mathbb{Z}_2^m\}
\end{tikzcd}$$

$$(F\phi)_{\mathbf{b}} = \frac{1}{\sqrt{2^m}} \sum_{\mathbf{a}\in\mathbb{Z}_2^m} (-1)^{\mathbf{b}\cdot\mathbf{a}} \phi_{\mathbf{a}}$$

- Dirac basis
- Eg. $e_{101} = (0,0,0,0,0,1,0,0)$

Waveforms/Sequences

$$\begin{tikzcd} H_D = \{D(0,\mathbf{b}) : \mathbf{b} \in \mathbb{Z}_2^m\} \arrow[r, "F"] & H_F = \{D(\mathbf{a},0) : \mathbf{a} \in \mathbb{Z}_2^m\} \end{tikzcd}$$

$$(F\phi)_{\mathbf{b}} = \tfrac{1}{\sqrt{2^m}} \textstyle\sum_{\mathbf{a}\in\mathbb{Z}_2^m} (-1)^{\mathbf{b}\cdot\mathbf{a}} \phi_{\mathbf{a}}$$

- The Fourier-Hadamard transform

$$F = \sqrt{\frac{1}{2^m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

interchanges H_D and H_F and their eigenvectors.

$\in \mathbb{Z}_2^m\}$

Waveforms/Sequences

$$\begin{tikzcd}
H_D = \{D(0,\mathbf{b}) : \mathbf{b} \in \mathbb{Z}_2^m\} \arrow[rr, "F"] \arrow[dd, dash] & & H_F = \{D(\mathbf{a},0) : \mathbf{a} \in \mathbb{Z}_2^m\} \arrow[dd, dash] \\
 & (F\phi)_{\mathbf{b}} = \frac{1}{\sqrt{2^m}} \sum_{\mathbf{a}\in\mathbb{Z}_2^m} (-1)^{\mathbf{b}\cdot\mathbf{a}} \phi_{\mathbf{a}} & \\
\{e_{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}_2^m\} \arrow[rr, "F"'] & & \{e_{\hat{\mathbf{b}}} = \frac{1}{\sqrt{2^m}} \sum_{\mathbf{a}\in\mathbb{Z}_2^m} (-1)^{\mathbf{b}\cdot\mathbf{a}} e_{\mathbf{a}} : \mathbf{b} \in \mathbb{Z}_2^m\}
\end{tikzcd}$$

- Walsh sequences
- $RM(1, m+1)$ as a linear code of length 2^m over $\mathbb{Z}_4$

Maximal Commutative Subgroups Corresponding to Unimodular Sequences

- Take any binary symmetric matrix P.
- Then we have the symplectic transform $U(P)$

$$U(P)D(\mathbf{a},\mathbf{b})U(P)^{\dagger} = i^{\mathbf{a}\cdot P\mathbf{a}}D(\mathbf{a},\mathbf{b}+P\mathbf{a})$$

- $\overline{H}_P = \{(\mathbf{a},P\mathbf{a}) : \mathbf{a}\in\mathbb{Z}_2^m\}$ is a maximal isotropic subspace.

$$\begin{tikzcd}
H_F = \{D(\mathbf{a},0) : \mathbf{a}\in\mathbb{Z}_2^m\} \arrow[r, "U(P)"] \arrow[d, dash] \arrow[dr, phantom, "(U(P)\phi)_{\mathbf{a}} = i^{\mathbf{a}\cdot P\mathbf{a}}\phi_{\mathbf{a}}"] & H_P = \{D(\mathbf{a},P\mathbf{a}) : \mathbf{a}\in\mathbb{Z}_2^m\} \arrow[d, dash] \\
\{\boldsymbol{e}_{\hat{\mathbf{b}}} = \frac{1}{\sqrt{2^m}}\sum_{\mathbf{a}\in\mathbb{Z}_2^m}(-1)^{\mathbf{b}\cdot\mathbf{a}}\boldsymbol{e}_{\mathbf{a}} : \mathbf{b}\in\mathbb{Z}_2^m\} \arrow[r, "U(P)"'] & \{\boldsymbol{e}_{P,\hat{\mathbf{b}}} = \frac{1}{\sqrt{2^m}}\sum_{\mathbf{a}\in\mathbb{Z}_2^m} i^{\mathbf{a}\cdot P\mathbf{a}}(-1)^{\mathbf{b}\cdot\mathbf{a}}\boldsymbol{e}_{\mathbf{a}} : \mathbf{b}\in\mathbb{Z}_2^m\}
\end{tikzcd}$$

Maximal Commutative Subgroups Corresponding to Unimodular Sequences

- Take any binary symmetric matrix P.
- Then we have the symplectic transform $U(P)$

$$U(P)D(\mathbf{a},\mathbf{b})U(P)^{\dagger} = i^{\mathbf{a}\cdot P\mathbf{a}}D(\mathbf{a},\mathbf{b}+P\mathbf{a})$$

- $\overline{H}_P = \{(\mathbf{a}, P\mathbf{a}) : \mathbf{a} \in \mathbb{Z}_2^m\}$ is a maximal isotropic subspace.

- $\mathfrak{F}_P$, ON basis of eigenvectors of H_P
- $\mathfrak{F}_P = \{D(\mathbf{a},\mathbf{b})\boldsymbol{e}_{P,\hat{\mathbf{0}}} \mid (\mathbf{a},\mathbf{b}) \in \mathbb{Z}_2^m \times \mathbb{Z}_2^m / H_P\}$
- Multidimensional binary chirps
- $RM(2, m+1)$ as a linear code of length 2^m over $\mathbb{Z}_4$
- Kerdock codes & Golay sequences

$(\mathbf{a}, P\mathbf{a}) : \mathbf{a} \in \mathbb{Z}_2^m\}$

$$\{\boldsymbol{e}_{\hat{\mathbf{b}}} = \tfrac{1}{\sqrt{2^m}} \textstyle\sum_{\mathbf{a}\in\mathbb{Z}_2^m} (-1)^{\mathbf{b}\cdot\mathbf{a}}\boldsymbol{e}_{\mathbf{a}} : \mathbf{b} \in \mathbb{Z}_2^m\} \xrightarrow[U(P)]{} \{\boldsymbol{e}_{P,\hat{\mathbf{b}}} = \frac{1}{\sqrt{2^m}} \sum_{\mathbf{a}\in\mathbb{Z}_2^m} i^{\mathbf{a}\cdot P\mathbf{a}}(-1)^{\mathbf{b}\cdot\mathbf{a}}\boldsymbol{e}_{\mathbf{a}} : \mathbf{b} \in \mathbb{Z}_2^m\}$$

Operators and Ambiguity Functions

- Any operator $S \in \mathfrak{D}$ can be represented as

$$S = \frac{1}{2^m} \sum_{\mathbf{a},\mathbf{b} \in \mathbb{Z}_2^m} \mathrm{Tr}(D(\mathbf{a},\mathbf{b})^\dagger S) D(\mathbf{a},\mathbf{b})$$

- The ambiguity function of a sequence ϕ

$$\mathcal{A}_\phi(\mathbf{a},\mathbf{b}) = \mathrm{Tr}(D(\mathbf{a},\mathbf{b})\Pi_\phi) = (\phi, D(\mathbf{a},\mathbf{b})\phi)$$

or

$$\Pi_\phi = \frac{1}{2^m} \sum_{\mathbf{a},\mathbf{b} \in \mathbb{Z}_2^m} \overline{\mathcal{A}_\phi(\mathbf{a},\mathbf{b})} D(\mathbf{a},\mathbf{b})$$

- Moyal's identity: $\mathrm{Tr}(\Pi_\phi \Pi_\psi) = |(\psi,\phi)|^2$ implies

$$\frac{1}{2^m} \sum_{\mathbf{a},\mathbf{b} \in \mathbb{Z}_2^m} |\mathcal{A}_\phi(\mathbf{a},\mathbf{b})|^2 = \| \phi \|^4$$

$\mathfrak{W}(\mathbb{Z}_2^m)$-Symmetry of Operators

- The *isotropy* subspace $\overline{H}_S \subset W(\mathbb{Z}_2^m)$ of an operator S is

$$\overline{H}_S = \{(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_2^m \times \mathbb{Z}_2^m | D(\mathbf{a},\mathbf{b})S = SD(\mathbf{a},\mathbf{b})\}$$

- If S has isotropy subspace $\overline{H}_S$ then

$$S = \sum_{(\mathbf{a},\mathbf{b})\in\overline{H}_S^{\perp}} s(\mathbf{a},\mathbf{b})D(\mathbf{a},\mathbf{b})$$

  where
  $\overline{H}_S^{\perp} \equiv \{(\mathbf{a},\mathbf{b}) \in \overline{H}_S | \langle(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')\rangle = 0 \; \forall(\mathbf{a}',\mathbf{b}') \in \overline{H}_S\}$.
- For $\phi \in \mathfrak{F}_P$, $\overline{H}_{\Pi_\phi} = \overline{H}_P = \{(\mathbf{a}, P\mathbf{a}) : \mathbf{a} \in \mathbb{Z}_2^m\}$
  - $\operatorname{supp} \mathcal{A}_\phi = \overline{H}_P$
  - $\mathcal{A}_\phi$ is unimodular on $\overline{H}_P$

Example: Kerdock Codes & Mutually Unbiased Bases

Dirac

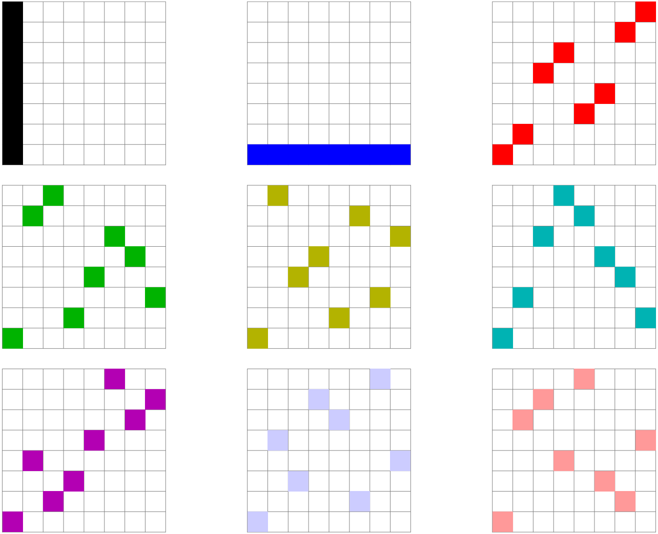
Fourier
Walsh
$RM(1,4)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

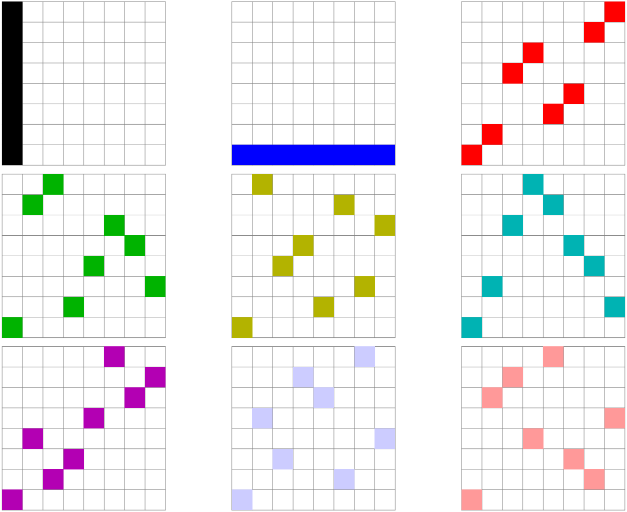
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

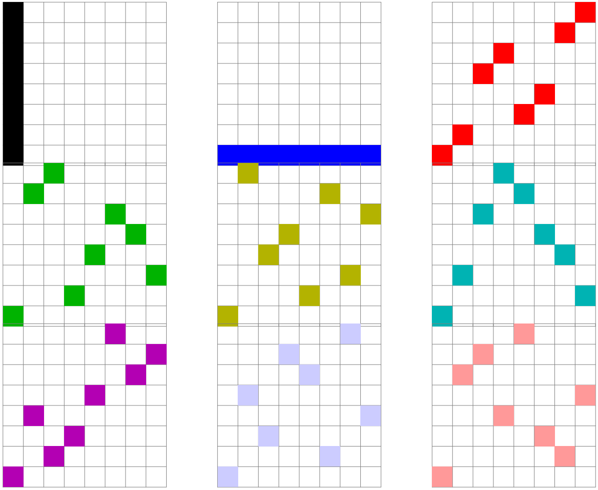
Example: Kerdock Codes & Mutually Unbiased Bases



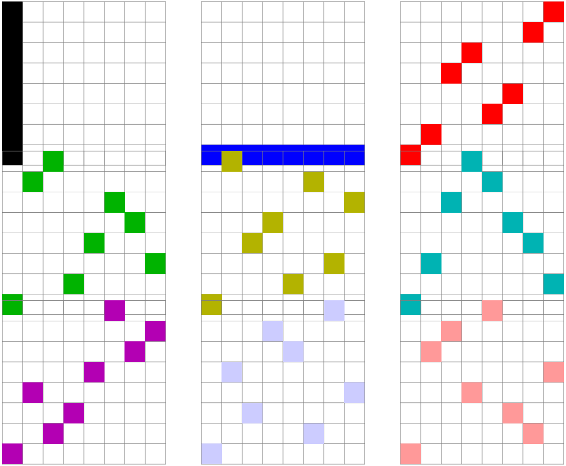
Example: Kerdock Codes & Mutually Unbiased Bases



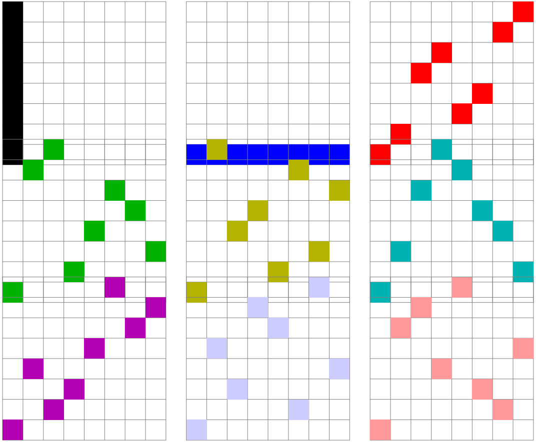
Example: Kerdock Codes & Mutually Unbiased Bases



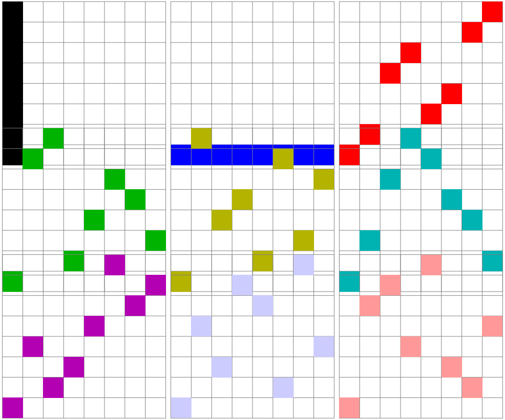
Example: Kerdock Codes & Mutually Unbiased Bases



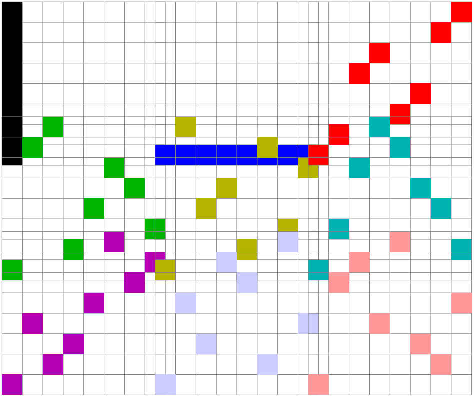
Example: Kerdock Codes & Mutually Unbiased Bases



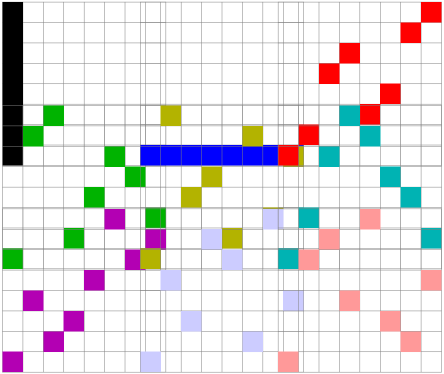
Example: Kerdock Codes & Mutually Unbiased Bases



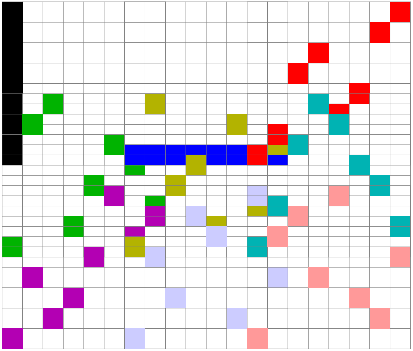
Example: Kerdock Codes & Mutually Unbiased Bases



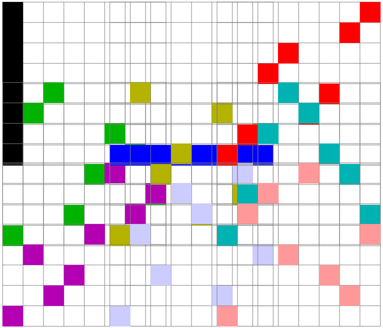
Example: Kerdock Codes & Mutually Unbiased Bases



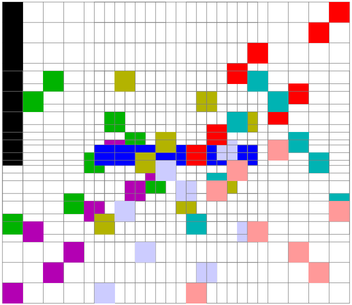
Example: Kerdock Codes & Mutually Unbiased Bases



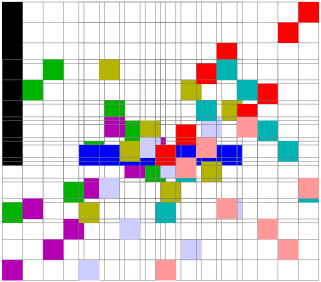
Example: Kerdock Codes & Mutually Unbiased Bases



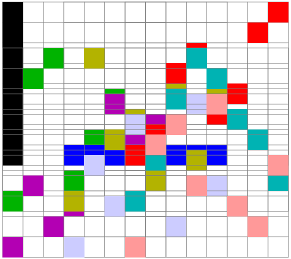
Example: Kerdock Codes & Mutually Unbiased Bases



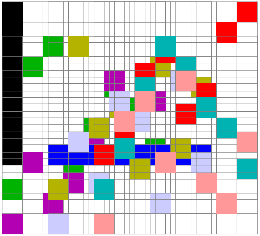
Example: Kerdock Codes & Mutually Unbiased Bases



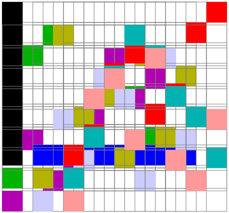
Example: Kerdock Codes & Mutually Unbiased Bases



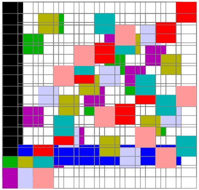
Example: Kerdock Codes & Mutually Unbiased Bases



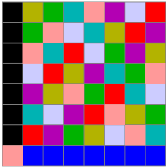
Example: Kerdock Codes & Mutually Unbiased Bases



Example: Kerdock Codes & Mutually Unbiased Bases

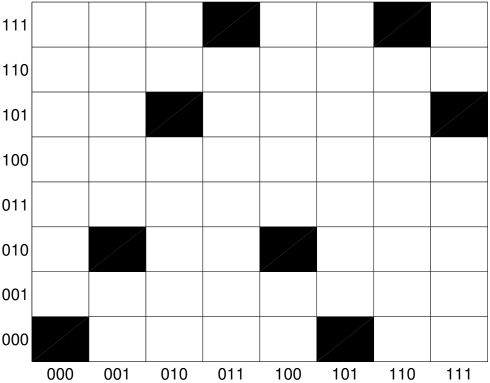


Example: Kerdock Codes & Mutually Unbiased Bases



Example: Classic Golay Sequences

Binary symmetric matrix $P_G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$



Example: Classic Golay Sequences (2)

- Fiducial vector:

$$e_{P_G,\hat{\mathbf{0}}} = (1/2)^{(3/2)} \sum_{\mathbf{a}\in\mathbb{Z}_2^3} (-1)^{a_1a_2+a_3a_2} e_{\mathbf{a}}$$

  In the Dirac basis $e_{P_G,\hat{\mathbf{0}}} = (1/2)^{(3/2)}(1,1,1,-1,1,1,-1,1)$
- For ϕ in $\mathfrak{F}_{P_G}$

$$\{\phi, D(000,100)\phi\} \quad \text{and} \quad \{\phi, D(000,001)\phi\}$$

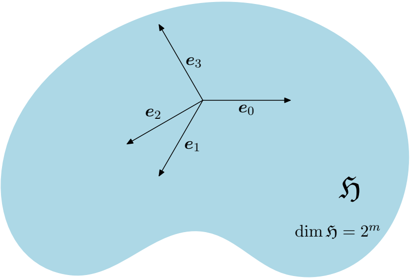
  are both Golay pairs.
- For example,

$$(1/2)^{(3/2)}(1,1,1,-1,1,1,-1,1)$$

  and

$$(1/2)^{(3/2)}(1,1,1,-1,-1,-1,1,-1)$$

Heisenberg-Weyl Group $\mathfrak{W}(\mathbb{Z}_{2^m})$

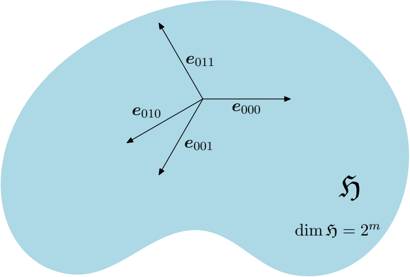


Unitary irreducible rep. of $\mathfrak{W}(\mathbb{Z}_{2^m})$

1. $\boldsymbol{\phi} = \sum_{\tau \in \mathbb{Z}_{2^m}} \phi_\tau \mathbf{e}_\tau$
2. $(\Delta(\tau, \nu)\boldsymbol{\phi})_{\tau'} = \omega^{\nu\tau'} \phi_{\tau' - \tau}$
3. For every linear operator on $\mathcal{H}$

$$S = \frac{1}{2^m} \sum_{\tau, \nu \in \mathbb{Z}_{2^m}} \mathrm{Tr}(S\Delta(\tau, \nu)^\dagger)\Delta(\tau, \nu)$$

Heisenberg-Weyl Group $\mathfrak{W}(\mathbb{Z}_2^m)$



Unitary irreducible rep. of $\mathfrak{W}(\mathbb{Z}_2^m)$

1. $\boldsymbol{\phi} = \sum_{\mathbf{a}\in\mathbb{Z}_2^m} \phi_{\mathbf{a}}\mathbf{e}_{\mathbf{a}}$
2. $(D(\mathbf{a},\mathbf{b})\boldsymbol{\phi})_{\mathbf{a}'} = (-1)^{\mathbf{b}\cdot\mathbf{a}'}\phi_{\mathbf{a}'\oplus\mathbf{a}}$
3. For every linear operator on $\mathcal{H}$

$$S = \frac{1}{2^m}\sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}_2^m} \mathrm{Tr}(SD(\mathbf{a},\mathbf{b})^\dagger)D(\mathbf{a},\mathbf{b})$$

Golay Complementary Sequences

Definition: *ϕ and ψ are $\mathbb{Z}_N$-Golay complementary if*

$$(\phi, \Delta(\tau, 0)\phi) + (\psi, \Delta(\tau, 0)\psi) = 0 \quad \textit{for } \tau \neq 0$$

This condition can be rewritten as

$$\mathrm{Tr}\left((\Pi_\phi + \Pi_\psi)\Delta(\tau, 0)\right) = 0 \quad \text{for } \tau \neq 0$$

Plan of Attack: Expand operators in terms of $D(\mathbf{a}, \mathbf{b})$ basis on the Hilbert Schmidt class of operators on $\mathcal{H}$

Support of the $\Delta(\tau, 0)$

- Let $C_{\mathbf{a}}$ denote the set of binary vectors covered by $\mathbf{a}$ and $\overline{\mathbf{w}} = (0, 1, 1, \cdots, 1)$ then

$$\Delta(\tau, 0) = \sum_{\mathbf{a} \in A_\tau} \frac{1}{|C_{\mathbf{a}} \cap C_{\overline{\mathbf{w}}}|} \sum_{\mathbf{b} \in C_{\mathbf{a}} \cap C_{\overline{\mathbf{w}}}} (-1)^{\mathbf{b} \cdot (\mathbf{c}_0(\mathbf{a}, \tau) \oplus \mathbf{a})} D(\mathbf{a}, \mathbf{b}),$$

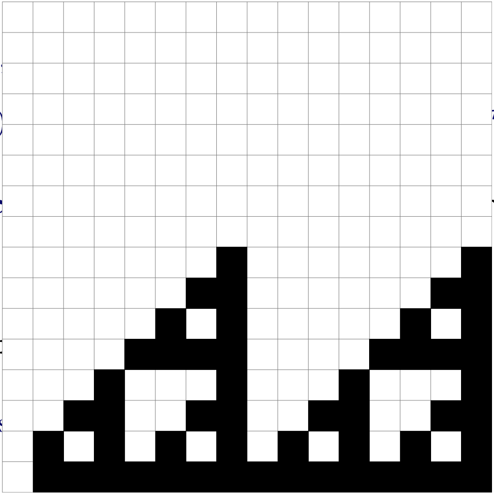
  where $\mathbf{c}_0(\mathbf{a}, \tau) \in C_{\mathbf{a}} \cap C_{\overline{\mathbf{w}}}$ is the unique solution of

$$\mathbf{c}_0 - \mathbf{c}_0 \oplus \mathbf{a} = \tau$$

- The support of the operators $\Delta(\tau, 0), \tau = 1, \cdots, 2^m$ is

$$\bigcup_{\tau=1}^{2^m-1} \operatorname{supp} \Delta(\tau, 0) = \{(\mathbf{a}, \mathbf{b}) : \mathbf{a} \in \mathbb{Z}_2^m \backslash \{\mathbf{0}\}, \mathbf{b} \in C_{\mathbf{a}} \cap C_{\overline{\mathbf{w}}}\}$$

Support of the $\Delta(\tau, 0)$



- Let $C_{\mathbf{a}}$ by $\mathbf{a}$ and
  $\overline{\mathbf{w}} = (0,$

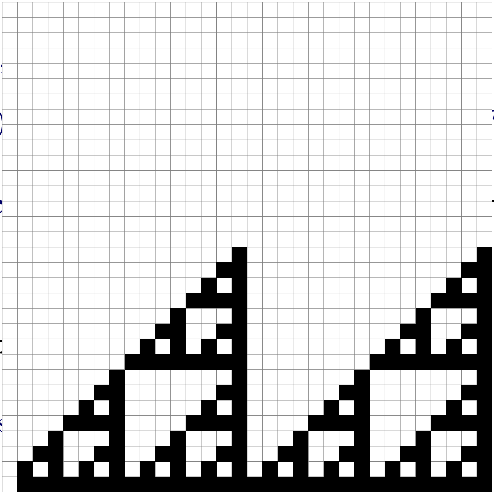
  $$\Delta(\tau, 0) \quad \tau)\oplus\mathbf{a}) \; D(\mathbf{a}, \mathbf{b}),$$

  where $\mathbf{c}$ of

- The sup $, 2^m$ is

  $$\bigcup_{\tau=1}^{2^m-1} \quad \in C_{\mathbf{a}} \cap C_{\overline{\mathbf{w}}}\}$$

Support of the $\Delta(\tau, 0)$



- Let $C_{\mathbf{a}}$ by $\mathbf{a}$ and $\overline{\mathbf{w}} = (0,$

  $\Delta(\tau, 0)$ $\tau)\oplus\mathbf{a})$ $D(\mathbf{a}, \mathbf{b}),$

  where $\mathbf{c}$ of

- The sup $, 2^m$ is

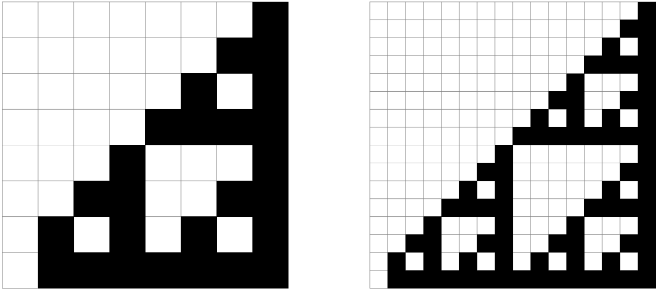
  $\bigcup_{\tau=1}^{2^m-1}$ s $\in C_{\mathbf{a}} \cap C_{\overline{\mathbf{w}}}\}$

Golay Complementarity on Orbits of $\mathfrak{W}(\mathbb{Z}_{2^m})$

Theorem

The pair of vectors $\{D(\mathbf{a},\mathbf{b})\phi, D(\mathbf{a},\mathbf{b})\psi\}$ *will be Golay complementary, for all* $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_2^m \times \mathbb{Z}_2^m$, *if and only if the support of* $\mathcal{A}_\phi + \mathcal{A}_\psi$ *does not intersect the set*

$$\mathcal{R}_\Delta = \{(\mathbf{a},\mathbf{b}) : \mathbf{a} \in \mathbb{Z}_2^m \backslash \{\mathbf{0}\}, \mathbf{b} \in C_\mathbf{a}\}$$



Golay Complementarity on Orbits of $\mathfrak{W}(\mathbb{Z}_{2^m})$

Theorem

The pair of vectors $\{D(\mathbf{a},\mathbf{b})\phi, D(\mathbf{a},\mathbf{b})\psi\}$ *will be Golay complementary, for all* $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_2^m \times \mathbb{Z}_2^m$*, if and only if the support of* $\mathcal{A}_\phi + \mathcal{A}_\psi$ *does not intersect the set*

$$\mathcal{R}_\Delta = \{(\mathbf{a},\mathbf{b}) : \mathbf{a} \in \mathbb{Z}_2^m \backslash \{\mathbf{0}\}, \mathbf{b} \in C_{\mathbf{a}}\}$$

$\mathcal{R}_\Delta$ is invariant under the symplectic transformations

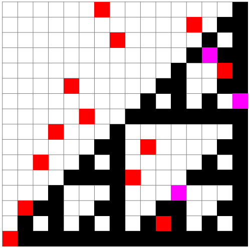
$$\begin{aligned} \mathbf{a} &\to \mathsf{S}\mathbf{a} \\ \mathbf{b} &\to \mathsf{S}\mathbf{b} \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbf{a} &\to \mathbf{a} \\ \mathbf{b} &\to \mathbf{b} + \mathsf{D}\mathbf{a} \end{aligned}$$

where S is a permutation and D is binary diagonal.

Why Maximal Commutative Subgroups?

- We want to minimise the support of $\mathcal{A}_\phi + \mathcal{A}_\psi$.
- The support of $\mathcal{A}_\phi$ is minimal if Π_ϕ has maximal isotropy subgroup H.
- The support of $\mathcal{A}_\phi + \mathcal{A}_\psi$ can be reduced even more if we choose $\psi \in \mathfrak{I}_H$, since then $\psi = D(\mathbf{a}_0, \mathbf{b}_0)\phi$ and

$$(\mathcal{A}_\phi + \mathcal{A}_\psi)(\mathbf{a}, \mathbf{b}) = (1 + (-1)^{\mathbf{a}_0 \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}_0})\mathcal{A}_\phi(\mathbf{a}, \mathbf{b})$$

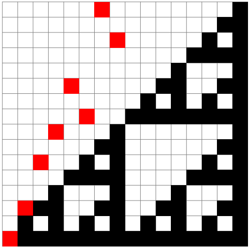


The choices $(\mathbf{a}_0, \mathbf{b}_0) = (000, 100)$ and $(\mathbf{a}_0, \mathbf{b}_0) = (000, 001)$ both remove the overlap.

Why Maximal Commutative Subgroups?

- We want to minimise the support of $\mathcal{A}_\phi + \mathcal{A}_\psi$.
- The support of $\mathcal{A}_\phi$ is minimal if Π_ϕ has maximal isotropy subgroup H.
- The support of $\mathcal{A}_\phi + \mathcal{A}_\psi$ can be reduced even more if we choose $\psi \in \mathfrak{F}_H$, since then $\psi = D(\mathbf{a}_0, \mathbf{b}_0)\phi$ and

$$(\mathcal{A}_\phi + \mathcal{A}_\psi)(\mathbf{a}, \mathbf{b}) = (1 + (-1)^{\mathbf{a}_0 \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}_0})\mathcal{A}_\phi(\mathbf{a}, \mathbf{b})$$

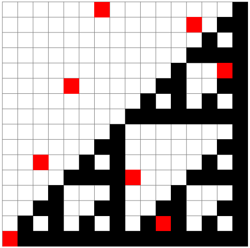


The choices
$(\mathbf{a}_0, \mathbf{b}_0) = (000, 100)$ and
$(\mathbf{a}_0, \mathbf{b}_0) = (000, 001)$ both
remove the overlap.

Why Maximal Commutative Subgroups?

- We want to minimise the support of $\mathcal{A}_\phi + \mathcal{A}_\psi$.
- The support of $\mathcal{A}_\phi$ is minimal if Π_ϕ has maximal isotropy subgroup H.
- The support of $\mathcal{A}_\phi + \mathcal{A}_\psi$ can be reduced even more if we choose $\psi \in \mathfrak{F}_H$, since then $\psi = D(\mathbf{a}_0, \mathbf{b}_0)\phi$ and

$$(\mathcal{A}_\phi + \mathcal{A}_\psi)(\mathbf{a}, \mathbf{b}) = (1 + (-1)^{\mathbf{a}_0 \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}_0})\mathcal{A}_\phi(\mathbf{a}, \mathbf{b})$$



The choices $(\mathbf{a}_0, \mathbf{b}_0) = (000, 100)$ and $\color{red}{(\mathbf{a}_0, \mathbf{b}_0) = (000, 001)}$ both remove the overlap.

Conclusion

- The Heisenberg-Weyl groups provide a basis for understanding the properties of many important waveforms, codes and sequences.
- Maximally commutative subgroups of the Heisenberg-Weyl groups, as well as their automorphism groups, the symplectic transformations, provide a key to this understanding.
- Important current areas of research in radar are the development of sequence/waveform libraries, which work collectively, and the development of multi-dimensional sequences/waveforms for transmission from antenna arrays.